GRADUATE MACROECONOMICS

UC Irvine

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(Partial and Incomplete)

"A barbarian is not aware that he is a barbarian."
– Jack Vance, *Big Planet*

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Quarter I

"We need no chieftain; such folk eat more than their share."
– Jack Vance, *Rhialto the Marvellous*

This section will introduce you to some benchmark models of the macroeconomy and to methods and tools that are commonly used in modern macroeconomics. It will cover models of economic growth that can also be used to study economic fluctuations, consumption, and investment. Most of the models will be dynamic and will include rational optimizing agents. The basic model of economic growth comes from Solow. While the model has no microfoundations, it is dynamic and it can serve as a building block for the neoclassical growth model. You will be presented with different extensions of this model that make the saving rate endogenous. You will first go over a model from Samuelson and Diamond where individuals live for a finite number of periods (for simplicity, two) and generations overlap. You will show that this simple model is very tractable and delivers important insights for welfare theorems in economies with an infinite time horizon. As a by-product, the model can also be used to study the role of fiat money and social security schemes. The second type of models you will be presented with has infinitely lived individuals. You will consider the problem of optimal economic growth from Ramsey. You will decentralize the Ramsey economy in order to obtain the neoclassical growth model with markets, households, and firms. Finally, you will go over models of endogenous growth. In terms of the methods, you will study dynamic systems (e.g., difference and differential equations, phase diagrams. . .) and dynamic optimization (optimal control theory and dynamic programming).

1.1. The Solow Growth Model

How can we explain the huge income differences across countries? A major paradigm began with Robert Solow and his contributions to the study of economic growth. The Solow model is a building block of modern macroeconomics and looks at the determinants of economic growth and the standard of living in the long run.

1.1.1. Solow's Model

- 4 variables: output (Y), capital (K), labor (L), and knowledge (A).
- A production function $F[\cdot]$ to link inputs and output.
- An equation for saving, investment, and capital accumulation.

Definition: The Neoclassical Production Function

$$Y(t) = F[K(t), A(t)L(t)]$$

- $Y(t)$ is the flow of output; 1 good that can be consumed or invested.
- $A(t)$ is labor augmenting technological progress that depends on time.
- $A(t)L(t)$ are efficiency units of labor.

Properties:

- Diminishing marginal products with respect to each input

$$\frac{\partial^2 F}{\partial K^2} < 0 \quad \text{and} \quad \frac{\partial^2 F}{\partial L^2} < 0.$$

- Constant returns to scale (C.R.S.) in its two arguments

$$F[cK, cAL] = cF[K, AL] \ \text{ for all } \ c \geq 0.$$

- The production function satisfies the Inada conditions

$$\lim_{K\to 0} F_K = \lim_{K\to 0} F_L = \infty$$
$$\lim_{K\to \infty} F_K = \lim_{K\to \infty} F_L = 0$$

1.1.2. Firm Optimization in the Solow Model

The firm's profit maximization problem is

$$\max_{\{K,L\}} F[K(t), A(t)L(t) - R(t)K(t) - w(t)L(t)].$$

- The rental price of capital is $R(t)$.
- The real wage is $w(t)$.

The first order conditions are

$$w(t) = A(t)F_L[K(t), A(t)L(t)]$$
$$R(t) = F_K[K(t), A(t)L(t)].$$

Theorem: Euler's Theorem
If $F(x, y)$ has constant returns to scale—it is homogenous of degree 1—then

$$F(x,y) = \frac{\partial F(x,y)}{\partial x}x + \frac{\partial F(x,y)}{\partial y}y.$$

So,

$$F[K(t), A(t)L(t)] = F_K K(t) + A(t)F_L L(t)$$
$$F[K(t), A(t)L(t)] = R(t)K(t) + w(t)L(t).$$

Note that real profits are

$$\pi = F[K(t), A(t)L(t)] - R(t)K(t) - w(t)L(t)$$
$$\pi = R(t)K(t) + w(t)L(t) - R(t)K(t) - w(t)L(t) = 0$$

Thus, payment of input factors exhausts profits.

1.1.3. The Production Function in Intensive Form

To write the production function in intensive form, normalize all the variables by the efficiency of labor supply $A(t)L(t)$. The production of one elementary unit of effective labor is

$$y(t) = F\left(\frac{K(t)}{A(t)L(t)}, 1\right) \equiv f\big(k(t)\big)$$

with effective capital

$$k \equiv \frac{K(t)}{A(t)L(t)}.$$

Properties:

- Monotonic: $f'(k) > 0$
- Concave: $f''(k) < 0$
- $\lim_{k\to 0}$ $f'(k) = +\infty$
- $\lim_{k\to +\infty}$ $f'(k) = 0$

The rental price of capital is

$$R(t) = f'[k(t)].$$

From Euler's theorem

$$f[k(t)] = k(t)R(t) + \frac{w(t)}{A(t)}.$$

Therefore, the real wage is

$$\frac{w(t)}{A(t)} = f[k(t)] - k(t)f'[k(t)].$$

Example: The Cobb–Douglas Specification

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha} \quad , \quad 0 < \alpha < 1$$

Written in intensive form

$$f(k) = F(k, 1)$$

$$f(k) = k^{\alpha}$$

This production function satisfies the 3 properties of a neoclassical production function; diminishing marginal returns, constant returns to scale, and the Inada conditions.

1.1.4. The Dynamics of the Solow Model

Labor grows at the rate n

$$\dot{L}(t) = nL(t) \Rightarrow L(t) = L(0)\mathrm{e}^{nt}.$$

Knowledge grows at the rate g

$$\dot{A}(t) = gA(t) \Rightarrow A(t) = A(0)\mathrm{e}^{gt}.$$

Agents save (invest) a fraction s of their income, while capital depreciates at rate δ

$$\dot{K}(t) = sY(t) - \delta K(t).$$

Definition: The saving rate, $s(\cdot)$, is the fraction of output that is saved.

The rate should depend on preferences for curent and future consumption, the level of wealth, the interest rate, etc. For simplification, s is assumed to be constant. This assumption matters for short–run dynamics and welfare results.

Take the log of $k = K/AL$

$$\ln k = \ln K - \ln A - \ln L$$

differentiate with respect to time

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - g - n$$

replace $\dot{K}/K$ to obtain

$$\frac{\dot{k}}{k} = \frac{sY}{K} - \delta - g - n.$$

Thus, the transitional equation for capital is

$$\dot{k} = sy - (\delta + g + n)k.$$

1.1.5. The Steady–State Equilibrium

Definition: An equilibrium of the Solow model is a function $k(t)$ that satisfies

$$\dot{k} = sf(k) - (\delta + g + n)k$$

with the initial condition $k(0) = k_0$.

Definition: The Balanced Growth Path
Note that $sf(k)$ is actual investment per unit of effective labor and $(\delta + g + n)k$ is breakeven investment. When actual investment equals breakeven investment, $\dot{k} = 0$, there is a steady state such that

$$sf(k) = (\delta + g + n)k.$$

- Since k is constant, K is growing at rate $n + g$.
- $Y = f(k)AL$ is also growing at rate $n + g$.
- Capital per worker $K/L = Ak$, output per worker $f(k)A$, and consumption per worker are growing at rate g.

Example: The Cobb–Douglas Specification
Production per efficient unit of labor is

$$f(k) = k^{\alpha}.$$

Steady–state capital per efficient unit of labor is

$$k^* = \left(\frac{s}{\delta + g + n}\right)^{\frac{1}{1-\alpha}}.$$

The conclusion is that countries that have high savings rates will tend to be richer and countries that have high population growth will tend to be poorer.

1.1.6. The Comparative Statics of the Solow Model

The effect from a change in the saving rate, s, on capital, k^*, is positive, found by differentiating $sf(k) = (\delta + g + n)$, it is

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{n + g + \delta - sf'(k^*)} = \frac{f(k^*)k^*}{s]f(k^*) - k^* f'(k^*)]} > 0.$$

The long run effect of a change in the saving rate, s, on output, y^*, is positive

$$\frac{\partial y^*}{\partial s} = f'(k^*)\frac{\partial k^*}{\partial s} = \frac{f'(k^*)f(k^*)k^*)}{s[f(k^*) - k^* f'(k^*)]} > 0.$$

The elasticity of output, y^*, with respect to the saving rate, s, is

$$\frac{s}{y}\frac{\partial y^*}{\partial s} = \frac{f'(k^*)k^*}{f(k^*) - f'(k^*)k^*}.$$

The effect from a change in a growth rate, n or g, on capital, k^*, is negative

$$\frac{\partial k^*}{\partial n} = \frac{\partial k^*}{\partial g} = \frac{-k^*}{n + g + \delta - sf'(k^*)} = \frac{-(k^*)^2}{s[f(k^*) - k^* f'(k^*)]} < 0.$$

Therefore,

$$\frac{s}{y}\frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1 - \alpha_K(k^*)} \quad \text{with} \quad \alpha + K(k^*) = \frac{k^* f'(k^*)}{f(k^*)}.$$

Example: Explaining Income Difference
Consider two countries, A and B. Let output per worker be $Af(k)$ and assume that

$$y_A = 10y_B \Rightarrow \ln y_A - \ln y_B = \ln 10.$$

Using the fact that

$$\alpha_K = \frac{\Delta \ln y}{\Delta \ln k}$$

then

$$\ln k_A - \ln k_B = \frac{\ln 10}{\alpha_K}.$$

If $\alpha = 1/3$, then the difference in k should be

$$\frac{k_A}{k_B} = 10^{\frac{1}{\alpha_K}} \simeq 10^3,$$

and there is no evidence of such differences in capital per worker, thus, differences in k cannot account for large differences in y.

Alternatively, consider the rate of return on capital (without depreciation)

$$r_K = f'(k).$$

Under a Cobb-Douglas specification

$$f(k) = k^{\alpha}.$$

So,

$$r_K = \alpha k^{\alpha-1} = \alpha y^{\frac{\alpha-1}{\alpha}}.$$

If $y_A = 10y_B$ and $\alpha = 1/3$, then $r_K \approx 1/3y^{-2}$ and

$$r_{K,B} = 100r_{K,A}.$$

There would be large incentives to invest in poor countries. Another way to explain differences in y is from differences in A. However, the growth of A is exogenous; A represents everything that we do not *know*.

1.1.7. The Golden–Rule of Capital Accumulation

Consumption per unit of effective labor is

$$c = f(k) - sf(k) = (1-s)f(k).$$

On the balanced growth path

$$c^* = f(k^*) - (n+g+\delta)k^*.$$

Thus

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - (n+g+\delta)]\frac{\partial k^*}{\partial s}.$$

Consumption is maximized when

$$\frac{\partial c^*}{\partial s} = 0.$$

This occurs when

$$f'(k^*) = n+g+\delta.$$

Definition: The golden–rule saving rate, s_{gold}, is the consumption–maximizing rate.

Definition: An economy is dynamically inefficient if $s > s_{\text{gold}}$. A reduction of the saving rate from s to s_{gold} would provide more consumption during the transition toward the new steady state, and more consumption at the steady state (i.e. the economy is oversaving: consumption could be raised at all points in time).

Suppose that $s < s_{\text{gold}}$. An increase of the saving rate would provide less consumption during the transition toward the new steady state, but more consumption at the steady state. Overall, the effect is positive if households do not care too much about current consumption. In the Solow growth model, there is nothing to guarantee that k will not be larger than the golden–rule level of capital. The saving decisions do not reflect intertemporal trade–offs.

1.1.8. The Transitional Dynamics of the Solow Model

How does the economy converge to its steady–state? Define

$$\gamma_k \equiv \frac{\dot{k}}{k} = s\frac{f(k)}{k} - (\delta + g + n).$$

Note that

$$\left[\frac{f(k)}{k}\right]' = \frac{f'(k)k - f(k)}{k^2} < 0.$$

Using l'Hopital's rule it can be shown

$$\lim_{k\to 0} \frac{f(k)}{k} = \lim_{k\to 0} f'(k) = \infty$$
$$\lim_{k\to \infty} \frac{f(k)}{k} = \lim_{k\to \infty} f'(k) = 0$$

If $k < k^*$, then $\gamma_k > 0$. The rate of growth of capital converges to 0 asymptotically. This implies that there are diminishing returns to capital. Similarly, an increase in s generates a positive effect on the growth rate of capital and output, but this effect is transitory. It can be shown

$$\frac{\partial \gamma_k}{\partial k} = s\frac{f'(k)k - f(k)}{k^2} < 0.$$

So

$$\frac{\partial \gamma_k}{\partial k} < 0.$$

Definition: Absolute convergence is that, other things equal, counties with a low capital stock per capita grow faster.

This concept suggests poor countries tend to *catch up*. For this to be true, countries must have the same s, n, g, δ, and $f(\cdot)$.

Definition: Conditional convergence is that an economy grows faster the further it is from its own steady–state value of capital.

Conditional convergence allows for heterogeneity across countries (i.e. countries may have different steady–states). Use

$$s = \frac{(\delta + g + n)k^*}{f(k^*)}$$

to rewrite γ_k as

$$\gamma_k = (\delta + g + n)\left[\frac{f(k)/k}{f(k^*)/k^*} - 1\right].$$

To determine how rapidly k approaches k^*, use a 1$^{\text{st}}$ order Taylor–series approximation

$$\dot{k} = [sf'(k^*) - (\delta + g + n)](k - k^*).$$

Let λ be defined as

$$\lambda = -[sf'(k^*) - (\delta + g + n)] > 0.$$

The solution to the 1$^{\text{st}}$ order linear differential equation is

$$k(t) - k^* = [k(0) - k^*]\exp(-\lambda t).$$

The speed of convergence depends on λ

$$\begin{aligned}
\lambda &= -[sf'(k^*) - (\delta + g + n)] \\
\lambda &= -\left[\frac{(\delta + g + n)k^*}{f(k^*)} f'(k^*) - (\delta + g + n)\right] \\
\lambda &= (\delta + g + n)[1 - \alpha_K(k^*)].
\end{aligned}$$

Let τ_{half} be the time required to be half–way between the initial capital stock and its steady–state value. Then

$$\frac{k(0) - k^*}{2} = [k(0) - k^*]\exp(-\lambda \tau_{\text{half}}) \Rightarrow \tau_{\text{half}} = \frac{\ln 2}{\lambda}.$$

1.1.9. The Dynamics with the Cobb-Douglas Production Function

- The production function is $f(k) = k^\alpha$.
- The law of motion for capital is

$$\dot{k} = sk^\alpha - (\delta + g + n)k.$$

- Define $x = k^{1-\alpha}$. The law of motion for $x(t)$ is

$$\dot{x} = (1 - \alpha)s - (1 - \alpha)(\delta + g + n)x.$$

The solution to the linear differential equation is

$$x(t) = \frac{s}{\delta + g + n} + \left(x(0) - \frac{s}{\delta + g + n}\right)\mathrm{e}^{-(1-\alpha)(\delta+g+n)t}.$$

Therefore, the path for capital accumulation is

$$k(t) = \left[\frac{s}{\delta + g + n} + \left(k_0^{1-\alpha} - \frac{s}{\delta + g + n}\right)\mathrm{e}^{-(1-\alpha)(\delta+g+n)t}\right]^{\frac{1}{1-\alpha}}.$$

The path for output accumulation is

$$y(t) = \left[\frac{s}{\delta + g + n} + \left(k_0^{1-\alpha} - \frac{s}{\delta + g + n}\right)\mathrm{e}^{-(1-\alpha)(\delta+g+n)t}\right]^{\frac{\alpha}{1-\alpha}}.$$

Definition: The rate of adjustment is $(1 - \alpha)(\delta + g + n)$.

A higher α means that there is less diminishing returns to capital, and hence a lower rate of adjustment. Similarly, a lower depreciation rate, δ, or a lower rate of technological progress, g, slows down the adjustment toward steady state.

The effects of an increase in the saving rate s are

- At any point in time, the capital stock and output increase.
- The path for consumption per efficient unit of labor is

$$c(t) = (1-2)\left[\frac{s}{\delta+g+n} + \left(k_0^{1-\alpha} - \frac{s}{\delta+g+n}\right)\mathrm{e}^{-(1-\alpha)(\delta+g+n)t}\right]^{\frac{\alpha}{1-\alpha}}.$$

- $c(t)$ decreases for low t.
- $c(t)$ increases for high t if $s < \alpha$.

1.1.10. The Discrete–Time Solow Growth Model

Many macro models are written in discrete time. Suppose there is no population growth ($n = 0$) and no technological progress ($g = 0$). The law of motion for the capital stock is

$$\begin{aligned} k_{t+1} &= k_t + i_t - \delta k_t \\ k_{t+1} &= (1-\delta)k_t + s y_t \\ k_{t+1} &= (1-\delta)k_t + s f(k_t) \end{aligned}$$

and is a first–order, nonlinear, difference equation.

There is an equilibrium (steady–state) such that

$$k_{t+1} = k_t = k^*.$$

The solution is

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}.$$

Since $f(k)/(k)$ is decreasing in k, there is a unique solution. Note that this is the same expression as in the continuous–time model.

All equilibria converge to the positive steady–state. To see this note

$$\frac{k_{t+1}-k_t}{k_t} = s\left[\frac{f(k_t)}{k_t} - \frac{\delta}{s}\right] = \left[\frac{f(k_t)}{k_t} - \frac{f(k^*)}{k^*}\right].$$

- If $k_t < k^*$, then $k_{t+1} > k_t$.
- If $k_t > k^*$, then $k_{t+1} < k_t$.
- Moreover,

$$k_{t+1} - k^* = [(1-\delta)k_t + sf(k_t)] - [(1-\delta)k^* + sf(k^*)]$$

  has the same sign as $k_t - k^*$.

Reading: Introduction and Section 1 of the paper titled "A Contribution to the Empirics of Economic Growth" by Mankiw, Romer and Weil (1992)

Reading: Chapter 2 and 3 from the book "Barriers to Riches" by Parente and Prescott (2000).

1.2. Linear, First–Order Differential Equations

1.2.1. Autonomous Equations

The general form of the linear, autonomous, first–order differential equation is

$$\dot{y} + ay = b$$

where

$$\dot{y} = \frac{\partial y(t)}{\partial t}$$

and a, b are known constants.

- Implicitly, y is a function of time t.
- Time is continuous ($t \in \mathbb{R}^+$).

Let $z(t)$ be a particular solution to this differential equation

$$\dot{z} + az = b.$$

Take the difference with the general equation

$$(\dot{y} - \dot{z}) + a(y - z) = 0$$

$$\dot{\widetilde{y}} + a\widetilde{y} = 0.$$

1.2.2. The Solution Method

- The first step is to find a solution $\widetilde{y}$ to the homogeneous form $\dot{\widetilde{y}} + a\widetilde{y} = 0$.
- The second step is to find a particular solution to the complete equation z.

The general solution is

$$y = \widetilde{y} + z.$$

1.2.3. The Homogeneous Solution

The homogeneous solution form is

$$\dot{y} + ay = 0$$

$$\text{where } a \neq 0.$$

$$\frac{\dot{y}}{y} = -a$$

$$\int \frac{\dot{y}}{y}\,\mathrm{d}t = -at + c_1.$$

Recall that

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln\big(f(x)\big) + c,$$

if we assume that $f(x) > 0$ for all x. Therefore,

$$\int \frac{\dot{y}}{y}\,\mathrm{d}t = -at + c_1 \Leftrightarrow \ln(y) + c_2 = -at + c_1$$

$$y = \exp[-at + (c_1 - c_2)]$$

$$y = C\exp[-at],$$

$$\text{where } C = e^{c_1 - c_2}.$$

Theorem: The general solution to the *homogeneous* form of the linear, autonomous, first–order differential equation is

$$\widetilde{y}(t) = C\mathrm{e}^{-at}.$$

Example:

$$\dot{y} - 3y = 0$$
$$\frac{\dot{y}}{y} = 3.$$

Integrate both sides

$$\ln y + c_2 = 3t + c_1.$$

Simplify and take the exponential

$$y(t) = C\mathrm{e}^{3t}.$$

1.2.4. The Particular Solution

When b is constant, a particular solution is the steady–state equilibrium value of y. A steady–state value $\overline{y}$ of a differential equation is defined by $\dot{y} = 0$.

$$\dot{y} + ay = b \underset{\dot{y}=0}{\Rightarrow} \overline{y} = \frac{b}{a}.$$

Theorem: The general solution to the *complete*, autonomous, linear, first–order differential equation is

$$y(t) = C\mathrm{e}^{-at} + \frac{b}{a}.$$

Example:

$$\dot{y} + 2y = 8.$$

In homogeneous form

$$\dot{\widetilde{y}} + 2\widetilde{y} = 0$$
$$\widetilde{y} = C\mathrm{e}^{-2t}.$$

A particular solution is

$$\dot{y} = 0 \Rightarrow \overline{y} = 4.$$

The general solution is

$$y(t) = \widetilde{y}(t) + \overline{y} = C\mathrm{e}^{-2t} + 4.$$

Example: Let $K(t)$ be the capital stock at time t. Capital depreciates at the rate $\boldsymbol{\Omega}$. Investment per unit of time is $\overline{I}$. The differential equation for capital is

$$\dot{K} = \overline{I} - \delta K.$$

The homogeneous form is

$$\dot{\widetilde{K}} = -\delta\widetilde{K}$$
$$\widetilde{K}(t) = C\mathrm{e}^{-\delta t}.$$

The steady–state solution is

$$\dot{K} = 0 \Rightarrow \overline{K} = \frac{\overline{I}}{\delta}.$$

The general solution is

$$K(t) = \widetilde{K}(t) + \overline{K} = C\mathrm{e}^{-\delta t} + \overline{K}$$

1.2.5. The Initial Value

In order to determine the constant C of the general solution, you need to know the value of y at some arbitrary time t_0. Assume

$$y(t_0) = y_0.$$

From the general solution

$$\begin{aligned} y_0 &= C\mathrm{e}^{-at_0} + \frac{b}{a} \\ C &= \mathrm{e}^{at_0}\left(y_0 - \frac{b}{a}\right). \end{aligned}$$

The solution for the differential equation becomes

$$\begin{aligned} y(t) &= \mathrm{e}^{at_0}\left(y_0 - \frac{b}{a}\right)\mathrm{e}^{-at} + \frac{b}{a} \\ y(t) &= \left(y_0 - \frac{b}{a}\right)\mathrm{e}^{-a(t-t_0)} + \frac{b}{a}. \end{aligned}$$

Example: Consider the example about capital accumulation and assume $K(0) = K_0$. From the general solution

$$\begin{aligned} K_0 \quad &= C + \overline{K} \Rightarrow C = K_0 - \overline{K} \\ K(t) &= (K_0 - \overline{K})\mathrm{e}^{-\delta t} + \overline{K}. \end{aligned}$$

1.2.6. Convergence

Does $y(t)$ converge to its steady–state value? Assume that $y(0) = y_0$. It follows that

$$y(t) = (y_0 - \overline{y})\mathrm{e}^{-at} + \overline{y}.$$

and

$$\lim_{t\to+\infty} y(t) = \overline{y} \Leftrightarrow \lim_{t\to+\infty} \mathrm{e}^{-at} = 0 \Leftrightarrow a > 0.$$

Theorem: The solution to a linear, autonomous, first–order differential equation, $y(t)$, converges to the steady–state equilibrium $\overline{y} = \frac{b}{a}$, no matter what the initial value, y_0, if and only if the coefficient in the differential equation is positive, $a > 0$.

1.3. Nonlinear, First–Order Differential Equations

1.3.1. Qualitative Analysis

Under which conditions can a solution to a nonlinear differential equation exist? Even if a solution exists, it is in general difficult to find. Usually qualitative analysis (e.g. phase diagrams) are useful.

Definition: The initial–value problem for an autonomous, nonlinear, first–order differential equation is expressed as

$$\begin{aligned} \dot{y} &= g(y) \\ y(t_0) &= y_0. \end{aligned}$$

Theorem: If the function g and its partial derivative $\frac{\partial g}{\partial y}$ are *continuous* in some closed rectangle containing the point (t_0, y_0), then in a neighborhood around t_0 contained in the rectangle, there is a unique solution, $y = \xi(t)$, satisfying

$$\begin{aligned} \dot{y} &= g(y) \\ y(t_0) &= y_0. \end{aligned}$$

Sometimes, you may not have an initial condition, but have a terminal condition or a transversality condition (e.g. the dynamics of the price of an asset).

Example: Consider the following nonlinear differential equation

$$\dot{y} = y - y^2 = y(1-y).$$

The steady–state values are

$$\dot{y} = 0 \Rightarrow y = 0 \quad \text{or} \quad y = 1.$$

The function $g(y) = y - y^2$ is concave and reaches a maximum for $y = 0.5$.

1.3.2. Phase Diagrams

Definition: A phase diagram shows $\dot{y}$ as a function of y.

It is useful to find the following.

- The range of y values of which y is *increasing* over time.
- The range of y values over which y is *decreasing* over time.
- Introduce arrows of motion to indicate the direction of motion of the variable y in the different regions.

Remember that the phase diagram for a difference equation plots y_{t+1} as a function of y_t. Steady–states are at the intersection of the phase line and the 45°degree line. The phase diagram for a differential equation plots the change in y, $\dot{y}$, as a function of y. Steady–states are at the intersection of the phase line and the horizontal axis.

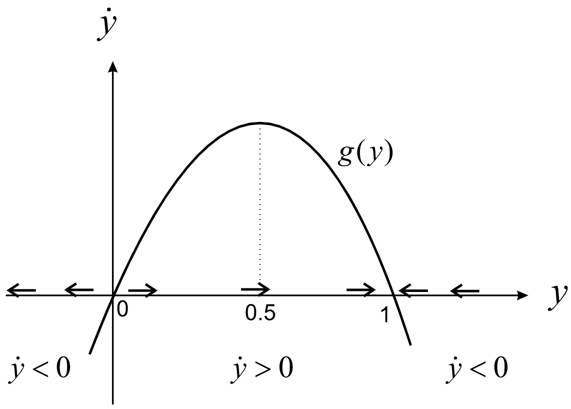


Figure I.1: Phase Diagram of a Differential Equation

1.3.3. Stability Analysis

To determine stability, you need to know the arrows of motion around the steady–state values. In Figure I.1, $y = 1$ is a stable equilibrium. The arrows of motion point toward the stable equilibrium and away from the unstable one.

1.3.4. Linearizing a Differential Equation

First, linearize $\dot{y} = g(y)$ in the neighborhood of the steady–state $\overline{y}$

$$g(y) = g(\overline{y}) + (y - \overline{y})g'(\overline{y})$$

where $g(\overline{y}) = 0$. The differential equation can be approximated by

$$\dot{y} = (y - \overline{y})g'(\overline{y}).$$

Thus

$$y(t) = C\mathrm{e}^{g'(\overline{y})t} + \overline{y}$$

implies convergence if $g'(\overline{y}) < 0$.

Theorem: A steady–state equilibrium point of a nonlinear, first–order differential equation is stable if the derivative $\frac{\mathrm{d}\dot{y}}{\mathrm{d}y}$ is negative at that point and unstable if the derivative is positive at that point.

Example: Consider again $\dot{y} = g(y) = y - y^2$.

$$\frac{\mathrm{d}\dot{y}}{\mathrm{d}y} = g'(y) = 1 - 2y.$$

At the steady–state value, $y = 0$,

$$\left.\frac{\mathrm{d}\dot{y}}{\mathrm{d}y}\right|_{y=0} = g'(0) = 1 > 0$$

and the equation is unstable.
At the steady–state value, $y = 1$,

$$\left.\frac{\mathrm{d}\dot{y}}{\mathrm{d}y}\right|_{y=1} = g'(1) = -1 < 0$$

and the equation is stable.

1.3.5. Interpretation

Assume the system is at its steady–state. It is pushed away from the equilibrium point by an amount $\mathrm{d}y$.

- If $\frac{\mathrm{d}\dot{y}}{\mathrm{d}y} < 0$, then the system will move backward and return to the equilibrium point.
- If $\frac{\mathrm{d}\dot{y}}{\mathrm{d}y} > 0$, then the system moves further away from equilibrium.

Example:

$$\dot{y} = 3y^2 - 2y = y(3y - 2)$$

The steady–state points, $\dot{y} = 0$, are

$$y = 0 \quad \text{or} \quad y = \frac{2}{3}.$$

The differential

$$\frac{\mathrm{d}\dot{y}}{\mathrm{d}y} = 6y - 2.$$

If $y = 0$, then

$$\left.\frac{\mathrm{d}\dot{y}}{\mathrm{d}y}\right|_{y=0} = -2 < 0$$

and the equation is stable.
If $y = \frac{2}{3}$, then

$$\left.\frac{\mathrm{d}\dot{y}}{\mathrm{d}y}\right|_{y=2/3} = 2 > 0$$

and the equation is unstable.

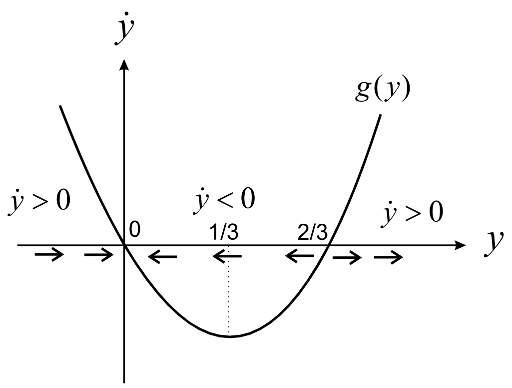


Figure I.2: Phase Diagram of a Differential Equation

1.3.6. The Neoclassical Model of Economic Growth

The production function, $Y = F(K, L)$, exhibits constant returns to scale. Output per person is

$$y \equiv \frac{Y}{L} = F\left(\frac{K}{L}, 1\right) \equiv f(k),$$

where $f(\cdot)$ is a concave function. The law of motion of the capital stock is

$$\dot{K} = sY.$$

The change in the capital-labor ratio is

$$\dot{k} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{K}{L}\right)$$
$$\dot{k} = \frac{\dot{K}}{L} - \frac{K\dot{L}}{L^2}$$
$$\dot{k} = \frac{\dot{K}}{L} - k\frac{\dot{L}}{L}.$$

The labor force grows at the constant rate, $\frac{\dot{L}}{L} = n$. It follows that

$$\dot{k} = \frac{sY}{L} - kn$$
$$\dot{k} = sf(k) - kn.$$

This nonlinear differential equation describes the growth of the economy.
The steady–state points, such that $\dot{k} = 0$, that occurs where

$$sf(k) = kn,$$

$$\text{are} \quad k = 0 \quad \text{and} \quad k^* > 0.$$

Furthermore,

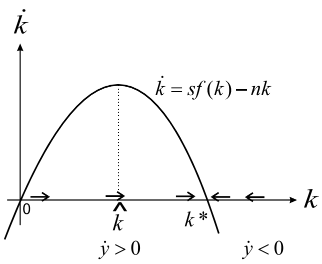
$$\frac{\mathrm{d}\dot{k}}{\mathrm{d}k} = sf'(k) - n$$
$$\frac{\mathrm{d}\dot{k}}{\mathrm{d}k} = 0 \Rightarrow sf'(k) = n$$

and this point is $\hat{k}$. If

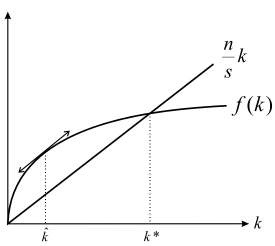
$$\frac{\mathrm{d}^2\dot{k}}{\mathrm{d}k^2} = sf''(k) < 0,$$

then $\dot{k}$ is maximized at $k = \hat{k}$.

Phase diagram



Steady-states



1.4. Growth in a Overlapping Generations Economy

The objective is to endogenize the savings rate in the Solow growth model by introducing a natural heterogeneity across individuals at a point in time. This is an example of an economy in which the competitive equilibrium may not be Pareto optimal. You can study the aggregate implications of life–cycle saving by individuals.

1.4.1. The OLG Model

Assumptions:

- Time is discrete ($t = 1, 2, \dots$)
- Each individual lives for two periods (the simplest case where generations overlap).
- L_t individuals are born in period t

$$L_t = (1+n)L_{t-1}$$

- At time 1, there is a generation who only lives for one period—the initial old—who own the initial capital stock.
- At any time, the economy is composed of 2 generations; the young and the old.
- Each individual supplies 1 unit of labor when young.
- Individuals are not productive when old.
- Capital saved in one period is a input in the production process of the following period.
- There is no depreciation of capital stock.

Households consume part of their first period income and save the rest to finance their second period retirement consumption. The capital stock is generated by individuals who save during their working lives. The timing of events is as follows

- 1st period of life: an individual is born, works, consumes, and saves capital.
- 2nd period of life: an individual spends revenue from capital, consumes, and dies.

The constant–relative–risk–aversion utility is

$$U_t = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta},$$

where $\theta > 0$, $\rho > -1$.

- $C_{1,t}$ is consumption in period t of young individuals.
- $C_{2,t+1}$ is consumption in period $t+1$ of old individuals.

$$C_{2,t+1} = (1 + r_{t+1})(w_t - C_{1,t}),$$

and the lifetime budget constraint is

$$C_{1,t} + \frac{C_{2,t+1}}{1 + r_{t+1}} = w_t.$$

The Lagrangian for the individual's maximization problem is

$$\max_{\{C_{1,t}, C_{2,t+1}\}} \mathscr{L} = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta} + \lambda\left[w_t - \left(C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}}\right)\right].$$

The first–order conditions are

$$\frac{\partial \mathscr{L}}{\partial C_{1,t}} = C_{1,t}^{-\theta} - \lambda = 0$$

and

$$\frac{\partial \mathscr{L}}{\partial C_{2,t+1}} = \frac{1}{1+\rho} C_{2,t+1}^{-\theta} - \frac{1}{1+r_{t+1}} \lambda = 0$$

So

$$C_{1,t}^{-\theta} = \lambda,$$

and the Euler equation is

$$C_{1,t}^{-\theta} = \frac{1+r_{t+1}}{1+\rho} C_{2,t+1}^{-\theta}.$$

You can note that

$$\frac{C_{1,t}}{C_{2,t+1}} = \left(\frac{1+r_{t+1}}{1+\rho} \right)^{\frac{1}{\theta}} \approx \frac{r_{t+1}-\rho}{\theta} - 1.$$

The interpretation of the Euler equation is

$$\frac{C_{2,t+1} - C_{1,t}}{C_{1,t}} \approx \frac{1}{\theta} \times (r_{t+1} - \rho)$$

"Rate of growth of consumption"

"Preference for consumption smoothing"

"Incentives to postpone consumption"

From the Euler equation and the budget constraint

$$C_{1,t} + C_{1,t} \frac{(1+r_{t+1})^{\frac{1}{\theta}-1}}{(1+\rho)^{\frac{1}{\theta}}} = w_t$$

and it follows that

$$C_{1,t} = [1 - s(r_{t+1})] w_t$$

with

$$s(r) = \frac{(1+r)^{\frac{1}{\theta}-1}}{(1+r)^{\frac{1}{\theta}-1} + (1+\rho)^{\frac{1}{\theta}}}.$$

- The saving rate, $s(r)$, is increasing in r if and only if $(1+r)^{\frac{1}{\theta}-1}$ is increasing in r. This occurs when $\theta < 1$.
- A rise in r has a negative substitution effect on current consumption.
- A rise in r has a positive income effect on current consumption (because the young agents are lenders of capital).
- If θ is low, then there is a high intertemporal elasticity of substitution and the substitution effect dominates.
- If $\theta = 1$, then it is the logarithmic case and the substitution effect and the income effect perfectly offset each other.

The capital stock in period $t+1$ is the amount saved by young individuals in period t

$$K_{t+1} = L_t s(r_{t+1}) w_t,$$

where L_t are the young in period t and s is the saving rate. If you divide by L_{t+1}

$$k_{t+1} = \frac{s(r_{t+1})}{(1+n)} w_t.$$

There are many firms, each with production function

$$Y_t = F(K_t, L_t).$$

Markets are competitive, so, labor and capital earn their marginal products

$$r_t = f'(k_t)$$
$$w_t = f(k_t) - k_t f'(k_t)$$

where $k_t \equiv \frac{K_t}{L_t}$. The initial capital stock K_0 is owned equally by all old individuals.

An equilibrium of the OLG model is a triple of sequences, $\{w_t\}$, $\{r_t\}$, $\{k_t\}$, that satisfy

$$r_t = f'(k_t)$$
$$w_t = f(k_t) - k_t f'(k_t)$$
$$k_{t+1} = \frac{s(r_{t+1})}{(1+n)} w_t$$

where k_0 is given.

1.4.2. The Steady–State and Dynamics of the OLG Economy

For examination of the dynamics for the capital stock, substitute r_{t+1} and w_t by their expressions as functions of the capital stock k_t

$$k_{t+1} = \frac{s[f'(k_{t+1})]}{(1+n)} [f(k_t) - k_t f'(k_t)].$$

The dynamics depends crucially on the saving rate function.

Next, with logarithmic utility and Cobb–Douglas production, $\theta = 1$, $f(k) = k^\alpha$, and $s(r) = \frac{1}{2+\rho}$, the saving rate is constant and independent of r, then

$$k_{t+1} = \frac{(1-\alpha)(k_t)^\alpha}{(2+\rho)(1+n)}.$$

The steady–state for k is

$$k_{t+1} = k_t = k^*$$
$$k^* = \left[\frac{(1-\alpha)}{(2+\rho)(1+n)}\right]^{\frac{1}{1-\alpha}}.$$

The steady–state capital stock is decreasing with ρ and n.

For examination of the speed of convergence, linearize the dynamic system around the balanced growth path

$$k_{t+1} \simeq k^* + \left.\frac{\mathrm{d}k_{t+1}}{\mathrm{d}k_t}\right|_{k_t=k^*} (k_t - k^*).$$

Then compute

$$\left.\frac{\mathrm{d}k_{t+1}}{\mathrm{d}k_t}\right|_{k_t=k^*} = \frac{\alpha(1-\alpha)(k^*)^{\alpha-1}}{(2+\rho)(1+n)} = \alpha.$$

The dynamic of the system can be approximated by

$$k_t - k^* = \alpha^t (k_0 - k^*).$$

The median lag satisfies

$$\frac{k_0 - k^*}{2} = \alpha^{\tau_{\text{half}}}(k_0 - k^*).$$

If $\alpha = 1/3$, then the median lag satisfies

$$\tau_{\text{half}} = \frac{\ln 2}{\ln 3}.$$

1.4.3. Dynamic Inefficiency in the OLG Economy

Next, we consider heterogenous agents and the questions of how to measure welfare and how to asses the efficiency of the equilibrium. A robust criterion is Pareto efficiency. Is it possible to raise consumption for all agents in all periods? If so, then the economy is dynamically inefficient. First, a feasible allocation satisfies

$$L_t C_{1,t} + L_{t-1} C_{2,t} + L_{t+1} k_{t+1} = L_t k_t + L_t f(k_t).$$

Now divide by L_t to yield

$$C_{1,t} + \frac{C_{2,t}}{1+n} + (1+n)k_{t+1} = k_t + f(k_t).$$

If the economy is in steady state, then

$$C_1 + \frac{C_2}{1+n} = f(k) - nk.$$

Definition: The golden–rule level of capital stock is where aggregate steady–state consumption is maximized.

Here, that is when

$$f'(k) = n.$$

If the steady–state equilibrium capital stock is larger than k_{GR}, then the economy is dynamically inefficient. That is, agents can "eat" the capital stock above k_{GR} and still increase aggregate consumption in subsequent periods. Consider a social planner and assume that $k^* > k_{\text{GR}}$. If the social planner does not change the capital stock, then the output available for consumption is $f(k^*) - nk^*$. The social planner can increase consumption in the current period and maintain the golden–rule level for the capital stock in subsequent periods. Consumption in the initial period is

$$f(k^*) - nk_{\text{GR}} + (k^* - k_{\text{GR}}),$$

and consumption in subsequent periods is

$$f(k_{\text{GR}}) - nk_{\text{GR}}.$$

The steady–state equilibrium capital stock is given by

$$k^* = \left[\frac{(1-\alpha)}{(2+\rho)(1+n)}\right]^{\frac{1}{1-\alpha}}.$$

The marginal product of capital is

$$f'(k^*) = \alpha {k^*}^{\alpha-1} = \frac{\alpha(2+\rho)(1+n)}{(1-\alpha)},$$

which may be greater than or less than $f'(k_{\text{GR}}) = n$. For sufficiently small α, then $f'(k^*) < f'(k_{\text{GR}})$. Equivalently, if the saving rate, $s = \frac{1}{2+\rho}$, is too large, then the steady–state capital stock exceeds the golden–rule level and the equilibrium is Pareto inefficient.

1.4.4. The Samuelson Paradox

Note that the 1^{st} Welfare Theorem states that competitive equilibria are always efficient. So, why does the 1^{st} Welfare Theorem fail to hold in the OLG model? This is because the 1^{st} Welfare Theorem assumes that there are no externalities, competitive markets, and no *missing* markets. In an economy with births and deaths, all agents cannot meet in a single market. A infinite number of dated commodities and a infinite number of agents, that is two infinite quantities, explains the Samuelson paradox.

Proof. Given infinite goods, g, and households, h, let $\{p_g\}$ be the competitive prices and $\{c_{hg}\}$ be the competitive allocation. The proof of the 1^{st} Welfare Theorem is sketched out as follows. Let $\{c'_{hg}\}$ be an allocation that Pareto–dominates the competitive allocation. It follows that

$$\sum_g p_g(c'_{hg} - c_{hg}) \geq 0,$$

for every h, with strict inequality for some h. Adding over over households yields

$$\sum_h \sum_g p_g(c'_{hg} - c_{hg}) > 0.$$

Interchanging the summations yields

$$\sum_g p_g \sum_h (c'_{hg} - c_{hg}) > 0,$$

which implies

$$\sum_h (c'_{hg} - c_{hg}) > 0,$$

for some good g. This violates feasibility; people are consuming more than their combined endowment of good g. However, this proof cannot be used in OLG models, because the sets of goods and agents are infinite. The proof of the 1^{st} Welfare Theorem requires the double summation to be finite. Thus, the 1^{st} Welfare Theorem does not hold in OLG economies. □

Example: Assume that there is no production, the endowment of the young is $e_1 = 1$, the endowment of the old $e_0 = 0$, and agents have linear utility

$$U_t = C_{1,t} + C_{2,t+1}.$$

The competitive equilibria are such that

$$(C_{1,t}, C_{2,t+1}) = (1, 0).$$

An allocation that generates a Pareto–improvement is

$$(C_{1,t}, C_{2,t+1}) = (0, 1),$$

because the old at time 1 are strictly better–off whereas the following generations are indifferent. Note that this is because generations are infinite, the young will always give their endowment to the old and receive an endowment when they are old. Everyone is just as well off, but the first generation of old is strictly better–off because they can consume an endowment.

1.4.5. Impure Altruism in the OLG Economy

So far, agents do not care about the utility of future generations. However, altruism might be empirically relevant, so, it is important to explain bequests. Perhaps, parents have warm glow preferences and derive utility from their bequests. We will examine how such altruism affects the dynamics of the OLG economy.

Assume that there are a continuum of individuals with measure normalized to 1 and that the population is constant at 1. Each individual lives for 2 periods; childhood and adulthood. In adulthood, the second period, each individual receives one child and an endowment of 1 unit of labor. Capital fully depreciates after use. Agents do not enjoy consumption in childhood, the first period. The preferences of agent i at time t are

$$U_i(t, c, b) = \log[c_i(t)] + \beta \log[b_i(t)],$$

where $c_i(t)$ is consumption when an adult and $b_i(t)$ is a bequest to the individual's offspring. The maximization problem is

$$\max_{c_i(t), b_i(t)} \log[c_i(t)] + \beta \log[b_i(t)]$$

$$\text{s. t. } c_i(t) + b_i(t) = w(t) + R(t) b_i(t-1),$$

where $R(t)$ is the rental price of capital and $w(t)$ is the real wage rate. Assuming competitve prices

$$\begin{aligned} R(t) &= f'[k(t)] \\ w(t) &= f[k(t)] - k(t) f'[k(t)]. \end{aligned}$$

The solution to this problem is

$$\begin{aligned} c_i(t) &= \frac{y_i(t)}{1+\beta} \\ b_i(t) &= \frac{\beta y_i(t)}{1+\beta}. \end{aligned}$$

The result is a distribution of wealth that evolves endogenously over time. The capital–labor ratio at time $t+1$ is given by aggregating the bequests of all adults at time t

$$\begin{aligned} k(t+1) &= \int_0^1 b_i(t) \,\mathrm{d}i \\ k(t+1) &= \int_0^1 \frac{\beta}{1+\beta} [w(t) + R(t) b_i(t-1)] \,\mathrm{d}i \\ k(t+1) &= \frac{\beta}{1+\beta} [w(t) + R(t) k(t)] \\ k(t+1) &= \frac{\beta}{1+\beta} f[k(t)]. \end{aligned}$$

This equation represents the aggregate equilibrium dynamics, and it is similar to the baseline Solow growth model. There is a unique positive steady–state, where capital stock increases with β,

$$k^* = \frac{\beta}{1+\beta} f(k^*).$$

At the steady–state, individual bequest dynamics are given by

$$b_i(t) = \frac{\beta}{1+\beta}[w^* + R^* b_i(t-1)],$$

and it can be checked that $\frac{R^*\beta}{1+\beta} < 1$. Thus, the distribution of bequests converges to full equality

$$b_i(t) \to b^* = \frac{\beta w^*}{1+\beta(1-R^*)}.$$

1.5. Fiat Money in the Overlapping Generations Economy

Definition: Fiat money is inconvertible, there is no promise it can be converted into anything else, and it is intrinsically useless, it cannot be used in the utility function nor in the production function.

Thus, fiat money is an efficient form of money; it can be produced at no cost. The Hahn problem is the question of how can an intrinsically useless object can have a positive value in exchange? This is a puzzle in monetary theory. The OLG model offers a "deep" model of money that can offer help to solve this puzzle by adding an inter-generational friction to motivate a meaningful role for money.

1.5.1. The OLG Barter Economy

First, look at the OLG barter economy.

Assumptions:

- Time is discrete; $t = 0, 1, 2, \dots$
- There are L_t individuals are born at time t

$$L_t = (1+n)^t.$$

- Individuals live for 2 periods.
- All agents have perfect foresight.
- Each agent is endowed with 1 unit of good when young.
- The good can be exchanged, consumed, or stored.
- Each unit saved at time t yields $1+r$ units at time $t+1$ (storage technology).
- The lifetime utility function of an individual born at time t is

$$u(c_{y,t}, c_{o,t+1}) = \ln(c_{y,t}) + \beta \ln(c_{o,t+1}),$$

  where $c_{y,t}$ is consumption when young and $c_{o,t+1}$ is consumption when old.

Let $\{(c_{y,t}, c_{o,t}), t = 0, 1, 2, \dots\}$ be an allocation of the consumption of the young and old agents in each period. When good are perishable, $r = -1$, feasible allocations satisfy

$$L_t c_{y,t} + L_{t-1} c_{o,t} \leq L_t$$

$$c_{y,t} + \frac{c_{o,t}}{1+n} \leq 1,$$

for all t. If agents face a similar budget constraint, then their program is

$$\max_{c_{y,t}, c_{o,t+1}} \ln(c_{y,t}) + \beta \ln(c_{o,t+1})$$

$$\text{s. t. } c_{y,t} + \frac{c_{o,t+1}}{1+n} = 1.$$

The solution (Euler equation) for the maximization problem is

$$\frac{c_{o,t+1}}{c_{y,t}} = \beta(1+n).$$

From the budget constraint, you can find that

$$c_{y,t} = \frac{1}{1+\beta}$$

$$c_{o,t+1} = \frac{\beta(1+n)}{1+\beta}$$

Even though the allocation $\{(\frac{1}{1+\beta}, \frac{\beta(1+n)}{1+\beta})\}$ is feasible, it is not attainable through bilateral trade. The young would like to exchange goods in this period against goods in the next period with the future young, but they can only trade with the current old. Therefore, no trade can take place and the decentralized outcome is $c_{y,t} = 1$ and $c_{o,t+1} = 0$. The decentralized equilibrium is not Pareto optimal and is therefore dynamically inefficient. If the young agents each transfer $\frac{\beta}{1+\beta}$ to the old generation and if each old agent receives $\frac{\beta(1+n)}{1+\beta}$, then there is a Pareto improvement and everyone is strictly better-off.

1.5.2. The OLG Monetary Economy

At time 0, the government gives to the old H divisible units of a fiat object called money. Suppose that at time t the price of goods in terms of this fiat object is P_t. Money is valued $P_t < +\infty$ or $\frac{1}{P_t} > 0$. The maximization problem of the agent is then

$$\max_{c_{y,t}, c_{o,t+1}} \ln(c_{y,t}) + \beta \ln(c_{o,t+1})$$

$$\begin{aligned} \text{s. t. } P_t c_{y,t} + m_t^d &= P_t \\ P_{t+1} c_{o,t+1} &= m_t^d, \end{aligned}$$

where m_t^d is the household's money balance. Let z_t be the real money balances

$$z_t = \frac{m_t^d}{P_t}.$$

The constraints become

$$\begin{aligned} c_{y,t} &= 1 - z_t \\ c_{o,t+1} &= \tfrac{P_t}{P_{t+1}} z_t, \end{aligned}$$

and the program can be rewritten

$$\max_{z_t} \ln(1 - z_t) + \beta \ln\left(\frac{P_t}{P_{t+1}} z_t\right),$$

with first order condition

$$\frac{1}{1 - z_t} = \frac{\beta}{z_t}.$$

The solution to this program is

$$\begin{aligned} c_{y,t} &= \frac{1}{1+\beta} \\ z_t &= \frac{\beta}{1+\beta} \\ c_{o,t+1} &= \frac{P_t}{P_{t+1}} \left(\frac{\beta}{1+\beta}\right). \end{aligned}$$

The equilibrium of the money market is

$$L_t P_t \left(\frac{\beta}{1+\beta}\right) = H,$$

where $L_t P_t(\frac{\beta}{1+\beta})$ is the *demand for money* and the H is the *money supply*. It follows that there is deflation (i.e. $P_t > P_{t+1}$) at the rate n

$$\frac{L_t P_t}{L_{t+1} P_{t+1}} = 1$$
$$\frac{P_t}{P_{t+1}} = 1 + n.$$

The allocation at the steady–state monetary equilibrium is

$$c_{y,t} = \frac{1}{1+\beta}$$
$$c_{o,t+1} = (1+n)\frac{\beta}{1+\beta}.$$

Thus, the introduction of money leads to a Pareto optimal allocation of resources across generations.

The assumption that the economy goes on forever is a necessary condition for money to be valued. If the economy ended at time T, the young at time T would not want to buy money. Proceeding backward, no one would ever want to buy money. Furthermore, even if a monetary equilibrium exists, there is also a barter equilibrium where fiat money is not valued. An implication of the incovertibility and intrinsic uselessness of fiat money is that equilibria in which fiat money is valued are tenuous.

1.5.3. The Role of Money in the OLG Economy

Now, suppose that there is fiat money in an economy with storage (i.e. $r > -1$). The amount of goods that is stored is k_t. The program of the agent is

$$\max_{c_{y,t}, c_{o,t+1}} \ln(c_{y,t}) + \beta \ln(c_{o,t+1})$$
$$\text{s. t. } c_{y,t} = 1 - k_t - z_t$$
$$c_{o,t+1} = (1+r)k_t + \frac{P_t}{P_{t+1}} z_t$$

The program can be rewritten as

$$\max_{k_t, z_t} \ln(1 - k_t - z_t) + \beta \ln\left((1+r)k_t + \frac{P_t}{P_{t+1}} z_t\right),$$

with first order conditions

$$-\frac{1}{c_{y,t}} + \beta\frac{(1+r)}{c_{o,t+1}} \begin{cases} \leq 0 \\ = 0 \quad \text{if } k_t > 0. \end{cases}$$
$$-\frac{1}{c_{y,t}} + \beta\frac{P_t/P_{t+1}}{c_{o,t+1}} \begin{cases} \leq 0 \\ = 0 \quad \text{if } z_t > 0. \end{cases}$$

From the Inada conditions, $c_{o,t+1} > 0$, which implies $k_t > 0$ or $z_t = \frac{m_t^d}{P_t} > 0$. From the FOC,

$$\text{If } \frac{P_t}{P_{t+1}} < 1 + r, \text{ then } k_t > 0 \text{ and } \frac{m_t^d}{P_t} = 0$$
$$\text{If } \frac{P_t}{P_{t+1}} > 1 + r, \text{ then } k_t = 0 \text{ and } \frac{m_t^d}{P_t} > 0.$$

If agents are willing to hold money, then

$$\frac{P_t}{P_{t+1}} = 1 + n.$$

The result is as follows.

- If $r < n$ then money can be valued and has a rate of return equal to n. The monetary economy achieves a Pareto optimum and storage is not used.
- If $r > n$, then the barter equilibrium is a Pareto optimum. There cannot be a monetary equilibrium with a constant money stock.

In conclusion, there can be a monetary equilibrium only if the barter equilibrium is not a Pareto optimum. In this case, there is a monetary equilibrium that is Pareto optimal. That is, if the economy is dynamically inefficient (i.e. $r < n$), then the introduction of money can make everybody better–off. Furthermore, money is valued only when it is not dominated in rate of return by any other asset.

1.5.4. Money and Inflation in the OLG Economy

Next, assume that the nominal money stock grows at rate σ

$$H_{t+1} = (1 + \sigma)H_t.$$

New money is introduced through lump–sum transfers to the old. Let T_t be the amount of the monetary transfer received by the old at time t. The program of an agent is

$$\max_{c_{y,t},c_{o,t+1}} \ln(c_{y,t}) + \beta \ln(c_{o,t+1})$$

$$\text{s. t. } c_{y,t} = 1 - k_t - z_t$$

$$c_{o,t+1} = (1 + r)k_t + \frac{P_t}{P_{t+1}} z_t + \frac{T_{t+1}}{P_{t+1}}.$$

The equilibrium condition in the money market is

$$L_t m_t^d = H_t,$$

or in real money balances it is

$$L_t z_t = \frac{H_t}{P_t}.$$

In a steady–state, per–capita real balances are constant

$$\frac{H_t}{L_t P_t} = \frac{H_{t+1}}{L_{t+1} P_{t+1}}.$$

This implies that

$$\frac{P_{t+1}}{P_t} = \frac{L_t H_{t+1}}{L_{t+1} H_t} = \frac{1 + \sigma}{1 + n}$$

where

$$\frac{1 + \sigma}{1 + n} \simeq 1 + \sigma - n.$$

The rate of return of money is

$$\frac{P_t}{P_{t+1}} - 1 = \frac{1 + n}{1 + \sigma} - 1 \simeq n - \sigma.$$

Assume that $\frac{1+n}{1+\sigma} > 1+r$. This implies that agents prefer to hold money rather than to store goods. Let z_t be the demand for real balances. The program of the agent is then

$$\max_{z_t} \ln(1-z_t) + \beta \ln\left(\frac{P_t}{P_{t+1}} z_t + \frac{T_{t+1}}{P_{t+1}}\right),$$

with first order condition

$$\frac{1}{1-z_t} = \frac{\beta}{z_t + \frac{T_{t+1}}{P_t}}.$$

The additional money is used to finance the transfer to the old

$$T_{t+1} = \frac{\sigma H_t}{L_t} = \sigma m_t^d,$$

or in real money balances

$$\frac{T_{t+1}}{P_t} = \sigma z_t.$$

It can then be deduced that

$$\begin{aligned} c_{y,t} &= \frac{1+\sigma}{1+\beta+\sigma} \\ z_t &= \frac{\beta}{1+\beta+\sigma} \\ c_{o,t+1} &= (1+n)\frac{\beta}{1+\beta+\sigma}. \end{aligned}$$

In conclusion, when real money balances are constant, then the price level is proportional to the money supply, that is, money is neutral. Money *does affect* the allocation of resources, that is, money is not superneutral. The monetary equilibrium with inflation is no longer a Pareto optimum. If $\frac{1+n}{1+\sigma} < 1+r$, then there is no monetary equilibrium (i.e. the storage technology outperforms money as a store of value). Also, note that money growth cannot be too large, otherwise the economy resorts to a barter economy.

1.5.5. Dynamics of the OLG Economy

Assume that goods are perishable (i.e. $r=-1$), the endowments are 1 when young and $\alpha < 1$ when old, and that the population is constant, $L_t = 1$ and $H = 1$. Consider the following utility function with no discounting

$$u(c_{y,t}, c_{o,t+1}) = \frac{(c_{y,t})^{1-\gamma_1}}{1-\gamma_1} + \frac{(c_{o,t+1})^{1-\gamma_2}}{1-\gamma_2},$$

where there are different coefficients for RRA across consumption, $\gamma_1, \gamma_2 > 0$. The program of the agent is

$$\max_{c_{y,t}, c_{o,t+1}} \frac{(c_{y,t})^{1-\gamma_1}}{1-\gamma_1} + \frac{(c_{o,t+1})^{1-\gamma_2}}{1-\gamma_2}$$

$$\begin{aligned} \text{s. t. } c_{y,t} + z_t &= 1 \\ c_{o,t+1} &= \frac{P_t}{P_{t+1}} z_t + \alpha. \end{aligned}$$

The FOC implies that

$$\frac{(c_{o,t+1})^{\gamma_2}}{(c_{y,t})^{\gamma_1}} = \frac{P_t}{P_{t+1}}.$$

By definition

$$z_{t+1} = z_t \frac{P_t}{P_{t+1}},$$

so, the budget constraints of the agent born at time t are

$$\begin{aligned} z_t &= 1 - c_{y,t} \\ \alpha + z_{t+1} &= c_{o,t+1} \end{aligned}$$

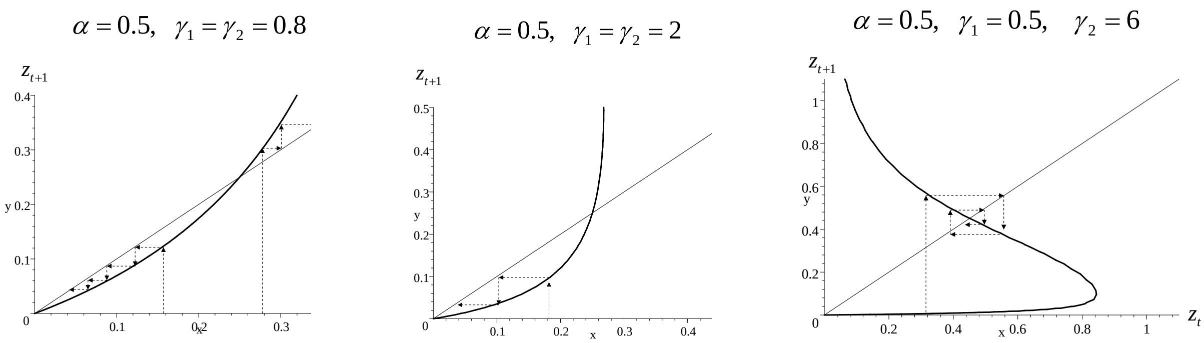
As a consequence

$$z_{t+1} = z_t \frac{(\alpha + z_{t+1})^{\gamma_2}}{(1 - z_t)^{\gamma_1}},$$

and it follows that

$$\frac{z_t}{(1 - z_t)^{\gamma_1}} = \frac{z_{t+1}}{(\alpha + z_{t+1})^{\gamma_2}}.$$

This represents a phase line. The left–hand side is increasing in z_t, and the right–hand side, $(\alpha(z_{t+1}^{-1/\gamma_2} + (z_{t+1})^{1-1/\gamma_2})^{-\gamma_2}$, is increasing in $z_t + 1$ if $\gamma < 1$. Otherwise, if $\gamma_2 > 1$, then the right–hand side is non-monotonic and the phase line may be backward bending.



The steady–states of this first–order difference equation are $\overline{z} = 0$ or $\overline{z}$ such that

$$(\alpha + \overline{z})^{\gamma_2} = (1 - \overline{z})^{\gamma_1},$$

where $\alpha < 1$ is required for a steady–state monetary equilibrium to exist.

When $\alpha = 0.5$ and $\gamma_1 = \gamma_2 = 0.8$, then there are two steady stats; $\overline{z} = 0$ and $\overline{z} = 0.25$. The monetary equilibrium is unstable. Paths that start to the right transition such that z increases and becomes larger than the initial endowment, 1, which is impossible. Path starting to the left transition such that z decreases asymptotically to 0. If you impose that the price level will not explode, then the monetary equilibrium is unique and the price level is uniquely determined. The monetary equilibrium will be locally stable. The economy converges to the monetary steady–state starting from any z in the neighborhood of $\overline{z}$. This implies that there are a multiplicity of convergent solutions, which in turn implies that the price level is indeterminate.

The reasons why the phase line can be backward bending are that the supply of goods when young depends on the rate of return of money, and if $\frac{P_t}{P_{t+1}}$ increases then there are two affects; a substitution effect, agents want to save more, and an income effect, agents want to consume more. The substitution effect may dominate for low values of $\frac{P_t}{P_{t+1}}$ whereas the income effect may dominate for large values of $\frac{P_t}{P_{t+1}}$.

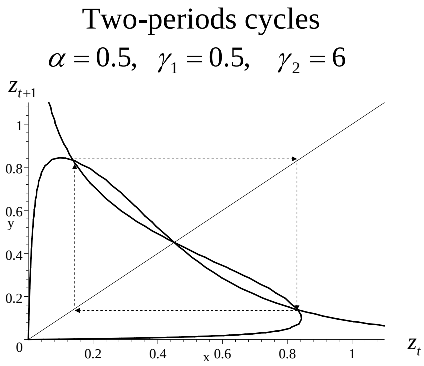
In order to find a cyclical solution, let the phase line by given by

$$z_{t+1} = \psi(z_t).$$

A 2–periods cycle is

$$z_2 = \psi(z_1) \text{ and } z_1 = \psi(z_2).$$

In order to find such a solution, you can draw the mirror image of the phase line, $z_t = \psi(z_{t+1})$, and check if it intersects the original phase line.



1.6. Introduction to Optimal Control Theory

1.6.1. A Simple Optimal Control Problem

There is one good and one agent (or social planner). The good can be either consumed or used as capital. The agent maximizes her lifetime discounted utility over the time horizon $[0, T]$. At time 0, there are $k_0 > 0$ units of capital. The terminal condition is that $k(T) = k_T$. The production function is

$$y = f(k)$$

where $f' > 0$, $f'' < 0$, $f'(0) = +\infty$, and $f'(+\infty) = 0$. Capital depreciates at rate $\delta > 0$. The law of motion for the capital stock is then

$$\dot{k} = f(k) - c - \delta k.$$

The utility of the agent is

$$U(\{c(t)\}) = \int_0^T \mathrm{e}^{-\rho t} u[c(t)] \, \mathrm{d}t,$$

where $\rho > 0$. $u(\cdot)$ is increasing and strictly concave; $u' > 0$, $u'' < 0$, $u'(0) = +\infty$, and $u'(+\infty) = 0$.

Definition: A state variable determines the position of the (economic) system at each point of time.

In this context, k is a state variable; $\{k(t), t \in [0, T]\}$ gives the trajectory of the system.

Definition: A control varialbe is the choice variable of the agent and affects her current utility and the path of the state variable(s).

In this context, c is a control variable.

The $\{c(t), t \in [0, T]\}$, initial condition $k(0) = k_0$, and the ordinary differential equation (ODE) for k define a unique trajectory for the system. A pair of functions, $c(t)$ and $k(t)$, that satisfy the law of motion for capital, the initial and terminal conditions, and non–negativity constraints is an admissible pair of functions.

The program of the agent is

$$\max_{\{c(t), k(t)\}} U(\{c(t)\}) = \int_0^T \mathrm{e}^{-\rho t} u[c(t)] \, \mathrm{d}t$$

$$\text{s. t.} \quad \dot{k} = f(k) - c - \delta k, \quad k(0) = k_0, \text{ and } \quad k(T) = k_T.$$

1.6.2. Discrete Time Optimal Control

Consider first a finite dimensional problem; a discrete problem where the number of control variables is finite. Divide the interval interval of time $[0, T]$ into N periods. The length of each period is $\Delta \equiv \frac{T}{N}$. The Agent's utility can be rewritten as

$$U(\{c(t)\}) = \sum_{\tau=0}^{N-1} \int_{\tau\Delta}^{(\tau+1)\Delta} \mathrm{e}^{-\rho t} u[c(t)] \, \mathrm{d}t.$$

At time $\tau\Delta$, with $\tau \in \{0, 1, \ldots, N-1\}$, the agent chooses $c_{\tau\Delta}$ given their utility over $[\tau\Delta, (\tau + 1)\Delta)$

$$U(\{c(t)\}) = \int_{\tau\Delta}^{(\tau+1)\Delta} \mathrm{e}^{-\rho t} u[c_{\tau\Delta}] \, \mathrm{d}t = u[c_{\tau\Delta}] \mathrm{e}^{-\rho\tau\Delta} \frac{1 - \mathrm{e}^{-\rho\Delta}}{\rho}.$$

If $\Delta \approx 0$, then $\mathrm{e}^{-\rho\Delta} \approx 1 - \rho\Delta$ and

$$\int_{\tau\Delta}^{(\tau+1)\Delta} \mathrm{e}^{-\rho t} u[c_{\tau\Delta}]\,\mathrm{d}t \approx \mathrm{e}^{-\rho\tau\Delta} u[c_{\tau\Delta}]\Delta.$$

Capital accumulation, over $[\tau\Delta, (\tau+1)\Delta]$, follows the law of motion for k

$$k_{(\tau+1)\Delta} - k_{\tau\Delta} = \int_{\tau\Delta}^{(\tau+1)\Delta} [f(k_t) - c_t - \delta k_t]\,\mathrm{d}t.$$

If Δ is small, the production flow can be approximated by

$$\int_{\tau\Delta}^{(\tau+1)\Delta} f(k_t)\,\mathrm{d}t \approx f[k_{\tau\Delta}]\Delta.$$

Depreciation of capital is approximated by

$$\int_{\tau\Delta}^{(\tau+1)\Delta} \delta k_t\,\mathrm{d}t \approx k_{\tau\Delta}\delta\Delta.$$

Therefore, the law of motion of k can be rewritten as

$$k_{(\tau+1)\Delta} = k_{\tau\Delta} + f(k_{\tau\Delta})\Delta - c_{\tau\Delta} - \delta\Delta k_{\tau\Delta}.$$

The agent's problem can be rewritten as

$$\max_{\{c_{\tau\Delta}, k_{(\tau+1)\Delta}\}} U(\{c(t)\}) = \sum_{\tau=0}^{N-1} \mathrm{e}^{-\rho\Delta\tau} u(c_{\tau\Delta})\Delta$$

$$\text{s. t.}\quad k_{(\tau+1)\Delta} = k_{\tau\Delta} + f(k_{\tau\Delta}) - c_{\tau\Delta} - \delta\Delta k_{\tau\Delta}, \quad k(0) = k_0, \text{ and } \quad k(T) = k_T.$$

Example: A Two–Period Optimal Control Problem
Suppose $N = 2$ and $\Delta = 1$. The program is

$$\begin{aligned} &\max_{\{c_0, c_1\}} \{u(c_0) + \mathrm{e}^{-\rho} u(c_1)\} \\ \text{s. t.}\quad & k_1 = k_0 + f(k_0) - c_0 - \delta k_0 \\ & k_2 = k_1 + f(k_1) - c_1 - \delta k_1 \end{aligned}$$

To solve, substitute c_1 and c_0 by their expressions given by the budget constraints. Now, the problem is

$$\max_{\{k1, k2\}} \{u[k_0 + f(k_0) - \delta k_0 - k_1] + \mathrm{e}^{-\rho} u[k_1 + f(k_1) - \delta k_1 - k_2]\}.$$

If k_2 is free, then $k_2 = 0$. Otherwise, k_2 is the terminal value. The first–order condition with respect to k_1 yields the Euler equation

$$u'(c_0) = [1 + f'(k_1) - \delta]\mathrm{e}^{-\rho} u'(c_1).$$

Example: A N-Period Optimal Control Problem
Let $\mu_{\tau\Delta}$ be the Lagrange multiplier associated with the law of motion k. Its economic interpretation is the shadow price of capital at time $\tau\Delta$. The Lagrangian is

$$\mathscr{L} = \sum_{\tau=0}^{N-1} \left\{ \mathrm{e}^{-\rho\Delta\tau} u(c_{\tau\Delta})\Delta + \mu_{\tau\Delta}\left[k_{\tau\Delta} + f(k_{\tau\Delta})\Delta - c_{\tau\Delta}\Delta - \delta\Delta k_{\tau\Delta} - k_{(\tau+1)\Delta}\right] \right\}.$$

Let $\lambda_{\tau\Delta} \equiv \mathrm{e}^{\rho\Delta\tau}\mu_{\tau\Delta}$ denote the current–value multiplier; the Lagrangian can be rewritten

$$\mathscr{L} = \sum_{\tau=0}^{N-1} \mathrm{e}^{-\rho\Delta\tau}\Big\{u(c_{\tau\Delta})\Delta + \lambda_{\tau\Delta}\big[k_{\tau\Delta} + f(k_{\tau\Delta})\Delta - c_{\tau\Delta}\Delta - \delta\Delta k_{\tau\Delta} - k_{(\tau+1)\Delta}\big]\Big\}.$$

The first–order conditions with respect to $c_{\tau\Delta}$ are

$$u'(c_{\tau\Delta}) = \lambda_{\tau\Delta} \quad \text{for all} \quad \tau = 0, \ldots, N-1.$$

The first–order conditions with respect to $k_{\tau\Delta}$ are

$$\lambda_{(\tau-1)\Delta}\mathrm{e}^{\rho\Delta} = \lambda_{\tau\Delta} + \lambda_{\tau\Delta}[f'(k_{\tau\Delta})\Delta - \delta\Delta] \quad \text{for all} \quad \tau = 1, \ldots, N.$$

Definition: The Hamiltonian function is defined as

$$H(c_{\tau\Delta}, k_{\tau\Delta}, \lambda_{\tau\Delta}) = u(c_{\tau\Delta}) + \lambda_{\tau\Delta}[f(k_{\tau\Delta}) - c_{\tau\Delta} - \delta k_{\tau\Delta}]$$

Definition: The multiplier of the Hamiltonian function is called a costate variable.

The first–order conditions with respect to $c_{\tau\Delta}$ are

$$\frac{\partial H}{\partial c_{\tau\Delta}} = 0 \quad \text{for all} \quad \tau = 0, \ldots, N-1.$$

The shadow price of capital obeys

$$\lambda_{(\tau-1)\Delta} = \mathrm{e}^{-\rho\Delta}\left(\frac{\partial H}{\partial k_{\tau\Delta}}\Delta + \lambda_{\tau\Delta}\right),$$

and the equation for the costate variable can be rewritten as

$$\left(\frac{\mathrm{e}^{\rho\Delta} - 1}{\Delta}\right)\lambda_{(\tau-1)\Delta} = \frac{\partial H}{\partial k_{\tau\Delta}} + \left(\frac{\lambda_{\tau\Delta} - \lambda_{(\tau-1)\Delta}}{\Delta}\right).$$

The solution is a system of two first–order difference equations. From the first first–order condition, you can express c as a function of λ. Therefore, there are two unknowns; the state variable k and the costate variable λ. An initial condition, $k(0) = k_0$, and a terminal condition, $k(T) = k_T$, are needed to pin down the trajectory of the system.

1.6.3. Continuous Time Optimal Control

For continuous time, take the limit as N goes to infinity (i.e. $\Delta \to 0$). Let Δ go to 0 while τ goes to infinity such as to maintain $\tau\Delta$ equal to t. The conditions

$$\frac{\partial H(c(t), k(t), \lambda(t))}{\partial c(t)} = 0 \quad \text{for all} \quad t \in [0, T]$$

and $k(T) = k_t$ remain unchanged. The difference equation for λ is a differential equation

$$\lim_{\Delta \to 0} \frac{\mathrm{e}^{\rho\Delta} - 1}{\Delta} = \rho$$

$$\lim_{\Delta \to 0} \frac{\lambda(t) - \lambda(t-\Delta)}{\Delta} = \lambda'(t).$$

The equation for the costate variable becomes

$$\rho\lambda = \frac{\partial H}{\partial k} + \frac{\partial \lambda}{\partial t}.$$

The candidates for an optimum satisfy the two first–order differential equations above.

It is assumed that the end–point is chosen freely by the agent. The boundary condition must hold

$$\lambda(T)k(T) = 0.$$

- If $k(T) > 0$, then the price of the state variable must be zero, $\lambda(T) = 0$, that is, the capital stock should be used until its marginal contribution is 0 at T.
- If $\lambda(T) > 0$, then $k(T) = 0$, that is, if capital is valuable, then the agent will die with no capital stock.

1.6.4. The Maximum Principle

Theorem: The Maximum Principle
The Maximum Principle states that an optimal solution to

$$\max_{\{c(t),k(t)\}} U(\{c(t)\}) = \int_0^T \mathrm{e}^{-\rho t} u[c(t)] \, \mathrm{d}t$$

$$\text{s. t.} \quad \dot{k} = f(k) - c - \delta k, \quad k(0) = k_0, \text{ and } \quad k(T) = k_T.$$

is a triplet $\{c(t), k(t), \lambda(t)\}$ that must satisfy the following conditions.

- There is optimal control, $c(t)$ maximizes $H\big(c(t), k(t), \lambda(t)\big)$ for any $t \in [0, T]$.
- The costate variable obeys the differential equation

$$\rho\lambda = \frac{\partial H}{\partial k} + \frac{\partial \lambda}{\partial t}.$$

- The terminal condition, $k(T) = k_T$ or $\lambda(T)k(T) = 0$ if the end–point is free, holds.

1.6.5. Sufficient Conditions for Optimal Control

The Maximum Principle only give the necessary conditions for an optimum. The first–order conditions may yield a local maximum, a minimum, or neither (i.e. a saddle–point). Some additional requirements are needed to isolate the trajectory(ies) that maximizes the welfare criterion.

Assumption: Assume that the Hamiltonian, $H(c, k, \lambda^*)$, is jointly concave in (c, k) where λ^* is generated by the Maximum Principle. The necessary conditions given by the Maximum Principle are sufficient conditions for a maximum. For the problem above, the Hamiltonian is

$$H(c, k, \lambda^*) = u(c) + \lambda^*[f(k) - c - \delta k],$$

where $\lambda^* \geq 0$. Thus, both the utility function and the production function must be strictly concave.

Proof. Consider a candidate solution $\{c^*(t), k^*(t), \lambda^*(t)\}$ that satisfies the first–order conditions. Now consider another admissible path $\{c(t), k(t)\}$. Denote $k^*(t)$ the trajectory under $c^*(t)$ and $k(t)$ if $c(t)$. Let Δ be defined as

$$\Delta = \int_0^T \mathrm{e}^{-\rho t} u[c^*(t)] \, \mathrm{d}t - \int_o^T \mathrm{e}^{-\rho t} u[c(t)] \, \mathrm{d}t.$$

Note that

$$u(c) = H(c, k, \lambda^*) - \lambda^* \dot{k}.$$

Substitute $u(c)$ by its expression given by H to get

$$\Delta = \int_0^T \mathrm{e}^{-\rho t}[H(c^*, k^*, \lambda^*) - H(c, k, \lambda^*)]\,\mathrm{d}t - \int_0^T \mathrm{e}^{-\rho t}\lambda^*[\dot{k}^* - \dot{k}]\,\mathrm{d}t.$$

Use the concavity of the Hamiltonian function in (c, k) to find

$$H(c, k, \lambda^*) \leq H(c^*, k^*, \lambda^*) + H_c(c^*, k^*, \lambda^*)(c - c^*) + H_k(c^*, k^*, \lambda^*)(k - k^*),$$

where H_c and H_k are the partial derivatives of the Hamiltonian with respect to c and k respectively. From the first order conditions

$$H_c(c^*, k^*, \lambda^*) = 0$$
$$H_k(c^*, k^*, \lambda^*) = \rho\lambda^* - \dot{\lambda}^*$$

you can obtain the following inequality

$$\Delta \geq \int_0^T \mathrm{e}^{-\rho t}[\rho\lambda^* - \dot{\lambda}^*](k^* - k)\,\mathrm{d}t - \int_0^T \mathrm{e}^{-\rho t}\lambda^*[\dot{k}^* - \dot{k}]\,\mathrm{d}t$$
$$\Delta \geq \int_0^T \frac{\mathrm{d}[-\mathrm{e}^{-\rho t}\lambda^*(k^* - k)]}{\mathrm{d}t}\,\mathrm{d}t$$
$$\Delta \geq \lambda^*(0)[k^*(0) - k(0)] - \mathrm{e}^{-\rho T}\lambda^*(T)[k^*(T) - k(T)].$$

Note that $k^*(0) = k(0) = k_0$ and $k^*(T) = k(T) = k_t$. Consequently,

$$\Delta \geq 0.$$

This inequality is strict if the Hamiltonian is strictly concave in (c, k). □

Theorem: The Mangasarian Sufficiency Theorem
If $(c^*(t), k^*(t))$ is a solution of the conditions provided by the Maximum Principle and if $H(c, k, \lambda^*)$ is concave in (c, k) with the costate variable, λ^*, supplied by the maximum principle, then $(c^*(t), k^*(t))$ solves the optimal control problem. If $H(c, k, \lambda^*)$ is strictly concave in (c, k), then $(c^*(t), k^*(t))$ is the unique solution to the problem.

1.6.6. Economic Interpretation of Optimal Control

The Hamiltonian of an agent is

$$H(c, k, \lambda) = u(c) + \lambda\dot{k}.$$

If the agent decides to modify her control variable, c, there are two consequences.

- First, the choice modifies the current utility of the agent.
- Second, the choice will affect the state variable, k, in future periods.

The question is, how to value this effect on $\dot{k}$? You can use a shadow price, λ, analogous to the Lagrange multiplier. The consequences of the current choice for the future are summarized by $\lambda\dot{k}$. The first first–order condition for the Hamiltonian is

$$\frac{\partial H(c, k, \lambda^*)}{\partial c} = 0$$

and as in a static problem, the agent chooses the control to maximize the objective. The second first–order condition for the Hamiltonian is

$$\rho\lambda = \frac{\partial H}{\partial k} + \frac{\partial \lambda}{\partial t}$$

and is like an asset–pricing equation. The left–hand side, $\rho\lambda$, can be interpreted as an opportunity cost. The first term on the right–hand side, $\frac{\partial H}{\partial k}$, is the dividend of the asset. The last term on the right–hand side, $\frac{\partial \lambda}{\partial t}$ is the capital gain or loss.

1.6.7. Costate Variables

The utility of the agent when the optimal control has been chosen is

$$U^* = \int_0^T u(c^*)\mathrm{e}^{-\rho t}\,\mathrm{d}t.$$

For any function, λ, you have

$$\lambda[f(k^*) - c^* - \delta k^*] = \lambda \dot{k}^*.$$

Consequently,

$$U^* = \int_0^T \mathrm{e}^{-\rho t}\{u(c^*) + \lambda[f(k^*) - c^* - \delta k^*] - \lambda \dot{k}^*\}\,\mathrm{d}t.$$

Integration by parts yields

$$\int_0^T \mathrm{e}^{-\rho t}\lambda k^*\,\mathrm{d}t = [\mathrm{e}^{-\rho t}\lambda k^*]_0^T - \int_0^T \mathrm{e}^{-\rho t}(-\rho\lambda + \dot{\lambda})k^*\,\mathrm{d}t$$
$$\int_0^T \mathrm{e}^{-\rho t}\lambda k^*\,\mathrm{d}t = \mathrm{e}^{-\rho T}\lambda(T)k^*(T) - \lambda(0)k^*(0) - \int_0^T \mathrm{e}^{-\rho t}(-\rho\lambda + \dot{\lambda})k^*\,\mathrm{d}t.$$

Substitute into the equation for U^* to obtain

$$U^* = \int_0^T \mathrm{e}^{-\rho t}\{H(c^*, k^*, \lambda) + (-\rho\lambda + \dot{\lambda})k^*\}\,\mathrm{d}t + \lambda(0)k^*(0) - \mathrm{e}^{-\rho T}\lambda(T)k^*(T)$$

Differentiate with respect to $k(0)$ to obtain

$$\frac{\partial U^*}{\partial k_0} = \int_0^T \mathrm{e}^{-\rho t}\left\{H_c(c^*, k^*, \lambda)\frac{\partial c^*}{\partial k_0} + [H_k(c^*, k^*, \lambda) - \rho\lambda + \dot{\lambda}]\frac{\partial k^*}{\partial k_0}\right\}\mathrm{d}t + \lambda(0) - \mathrm{e}^{-\rho T}\lambda(T)\frac{\partial k^*(T)}{\partial k_0}.$$

Note that the shadow price, λ, is an arbitrary function of time. A change in the initial condition, $k(0)$, does not affect the costate variable, λ, or its derivative. Furthermore, $\frac{\partial k^*(T)}{\partial k_0} = 0$, because the terminal point is exogenously specified. Suppose that you select the optimal path $\lambda^*(t)$ from the Maximum Principle. Then

$$\frac{\partial U^*}{\partial k_0} - \lambda^*(0).$$

Note that $\lambda^*(0)$ measures the impact of a change in the initial capital stock on the utility of the agent. The effect of a change in k_T on the agent's utility is

$$\frac{\partial U^*}{\partial k_T} = -\mathrm{e}^{-\rho T}\lambda^*(T).$$

1.6.8. Infinite Time Horizon Optimal Control

When the time horizon is infinite, the method is similar. The Hamiltonian is

$$H(c, k, \lambda) = u(c) + \lambda[f(k) - c - \delta k].$$

The optimal control satisfies

$$\frac{\partial H}{\partial c} = 0.$$

The dynamic equation for the costate variable satisfies

$$\rho\lambda = \frac{\partial H}{\partial k} + \dot{\lambda}.$$

There is an initial condition $k(0) = k_0$. However, there is no terminal value for the capital stock. Instead, there is a transversality condition.

Definition: The Transversality Condition
Assume that $H(c, k, \lambda)$ is jointly concave in c and k. Let c^* and k^* be the optimal paths for consumption and the capital stock, and let λ^* be the associated costate variable. Consider another path, c and k, that satisfies

$$\begin{aligned} \dot{k} &= f(k) - c - \delta k \\ k(0) &= k_0. \end{aligned}$$

The path (c^*, k^*) is an optimum if and only if

$$\Delta = \int_0^{+\infty} \mathrm{e}^{-\rho t} u[c^*(t)\, \mathrm{d}t - \int_0^{+\infty} \mathrm{e}^{-\rho t} u[c(t)]\, \mathrm{d}t \geq 0.$$

Note that

$$\begin{aligned} u(c^*) &= H(c^*, k^*, \lambda^*) - \lambda^* \frac{\partial k^*}{\partial t} \\ u(c) &= H(c, k, \lambda^*) - \lambda^* \frac{\partial k}{\partial t} \end{aligned}$$

From the concavity of the Hamiltonian, you have

$$H(c^*, k^*, \lambda^*) - H(c, k, \lambda^*) \geq H_k(c^*, k^*, \lambda^*)(k^* - k) + H_c(c^*, k^*, \lambda^*)(c^* - c).$$

From the first–order conditions

$$\begin{aligned} H_c(c^*, k^*, \lambda^*)(c^* - c) &= 0 \\ H_k(c^*, k^*, \lambda^*)(k^* - k) &= (\rho\lambda^* - \dot{\lambda}^*)(k^* - k) \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^{+\infty} \mathrm{e}^{-\rho t}[u[c^*(t)] - u[c(t)]]\, \mathrm{d}t &\geq \int_0^{+\infty} \mathrm{e}^{-\rho t}(\rho\lambda^* - \dot{\lambda}^*)(k^* - k)\, \mathrm{d}t - \int_0^{+\infty} \mathrm{e}^{-\rho t} \lambda^* \left(\frac{\partial k^*}{\partial t} - \frac{\partial k}{\partial t} \right) \mathrm{d}t \\ &\geq \int_o^{+\infty} -\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-\rho t}\lambda^*(k^* - k)]\, \mathrm{d}t \\ &\geq \lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda^*(k - k^*). \end{aligned}$$

A sufficient condition for (c^*, k^*) to be a maximum is that

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda^*(k - k^*) \geq 0 \quad \text{for all } k.$$

In this problem, the shadow price of capital will always be positive and the capital stock cannot be negative. Consequently, a sufficient condition for an optimum is that

$$\boxed{\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda^* k^* = 0}$$

Theorem: The Mangasarian Sufficiency Theorem
Let (c^*, k^*, λ^*) be a triplet generated by the Maximum Principle. If $H(c, k, \lambda^*)$ is jointly concave in k and c, and if $lim_{t\to\infty}\mathrm{e}^{-\rho t}\lambda^*(k - k^*) \geq 0$ for all possible paths, $k(t)$, then (c^*, k^*) is optimal.

1.6.9. Generalization of the Optimal Control Problem

For a more general problem, $x(t)$ is the state variable and $y(t)$ is the control variable. The optimal control problem is

$$\max_{\{x(t),y(t)\}} \int_0^\infty u[t, x(t), y(t)]\, \mathrm{d}t$$

subject to

$$\dot{x}(t) = g[t, x(t), y(t)],$$

an initial condition, $x(t) = x_0$, and $\lim_{t\to\infty} b(t)x(t) \geq x_1$ with $\lim_{t\to\infty} b(t) < \infty$.

In the previous example, $x(t) = k(t)$, $y(t) = c(t)$, $u(t, x, y) = \mathrm{e}^{-\rho t}u(c)$, and $g(t, x, y) = f(x) - \delta x - y$. However, with an infinite horizon, then $\lim_{t\to\infty} b(t)x(t) \geq x_1$ is a terminal value constraint. In many applications, $b(t) = 1$.

Definition: Value Function
The optimal value of the dynamic maximization problem starting at time t_0 with state variable $x(t_0)$ is given by the value function

$$V[t_0, x(t_0)] = \max_{\{x(t),y(t)\}} \int_{t_0}^\infty u[t, x(t), y(t)]\, \mathrm{d}t$$

$$\text{s. t.} \quad \dot{x}(t) = g[t, x(t), y(t)], \quad x(t) = x_0, \quad \text{and} \quad \lim_{t\to\infty} b(t)x(t) \geq x_1.$$

Theorem: The Principle of Optimality
Suppose that $(x^*(t), y^*(t))$ is a solution to the dynamic optimization problem. Then

$$V[t_0, x(t_0)] = \int_{t_0}^{t_1} u[t, x^*(t), y^*(t)]\, \mathrm{d}t + V[t_1, x^*(t1)],$$

for all $t_1 \geq t_0$.

Proof. Assuming that $(x^*(t), y^*(t))$ is a solution to the optimization problem implies

$$V[t_0, x(t_0)] = \int_{t_0}^{\infty} u[t, x^*(t), y^*(t)]\, \mathrm{d}t$$

$$V[t_0, x(t_0)] = \int_{t_0}^{t_1} u[t, x^*(t), y^*(t)]\, \mathrm{d}t + \int_{t_1}^{\infty} u[t, x^*(t), y^*(t)]\, \mathrm{d}t.$$

By definition of the value function

$$V[t_1, x^*(t_1)] \geq \int_{t_1}^{\infty} u[t, x^*(t), y^*(t)]\, \mathrm{d}t.$$

Thus, it is clear that the inequality cannot be strict, otherwise there would be a profitable deviation after t_1. □

Theorem: The Infinite–Horizon Maximum Principle
Suppose that the dynamic maximization problem has a solution $(x^*(t), y^*(t))$. Define the present value Hamiltonian as

$$H(t, x, y, \lambda) = u(t, x, y) + \lambda g(t, x, y).$$

Then

$$y^*(t) \in \arg\max_{\{y\}} H[t, x^*(t), y, \lambda(t)] \quad \text{for all } t.$$

$$\dot{\lambda}(t) = -H_x[t, x^*(t), y^*(t), \lambda(t)].$$

Example: Optimal Growth
The Hamiltonian is

$$H(t, k, c, \lambda) = \mathrm{e}^{-\rho t} u(c) + \lambda[f(k) - \delta k - c].$$

The maximization of H with respect to c yields

$$u'(c(t)) = \mathrm{e}^{\rho t} \lambda(t).$$

The differential equation for the costate variable is

$$\dot{\lambda}(t) = -\lambda(t)[f'(k) - \delta].$$

Denote $\mu(t) \equiv \mathrm{e}^{\rho t} \lambda(t)$. Then,

$$\rho \mu(t) = \mu(t)[f' - \delta] + \dot{\mu}(t).$$

1.7. The Ramsey Model of Optimal Growth

This section will introduce you to a growth model from Ramsey 1928. The question is, how much should a nation save? A framework for studying the optimal intertemporal allocation of resources is introduced. The model begins with microfoundations, wherein the optimizing behavior of agents is explicit, and the result is that the saving rate is endogenous.

1.7.1. The Ramsey Model

Assumptions:

- There is a household composed of L agents who have to decide how much to consume and how much to save (invest).
- Output is produced according to a Neoclassical production function

$$Y = F(K, L).$$

- There is initial capital, k_0.
- Capital depreciates at rate δ.

For simplicity, you can normalize L to 1, and work in per capita terms. Let

$$f(k) \equiv F(k, 1).$$

The function $f(\cdot)$ is strictly concave and satisfies the Inada conditions;

$$f(0) = 0, \quad f'(0) = \infty, \quad f'(\infty) = 0.$$

The household's utility function is

$$U = \int_0^{+\infty} \mathrm{e}^{-\rho t} u\big(c(t)\big)\,\mathrm{d}t,$$

where ρ is the rate of time preference. The instantaneous utility function is

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \quad \text{where } \theta > 0$$

and

$$u(c) = \ln c \quad \text{if } \theta = 1.$$

Definition: The Coefficient of Relative Risk Aversion

$$\text{RRA} \equiv \frac{-cu''(c)}{u'(c)} = \theta.$$

Definition: The Intertemporal Elasticity of Substitution

$$\eta = \frac{1}{\theta}.$$

Proof. The utility can be rewritten

$$U = u[c(t)] + \mathrm{e}^{-\rho(s-t)} u[c(s)].$$

Thus the marginal rate of substitution is

$$-\left.\frac{\mathrm{d}c(t)}{\mathrm{d}c(s)}\right|_{U=cste} = \frac{u'[c(s)]\mathrm{e}^{-\rho(s-t)}}{u'[c(t)]}$$

$$\text{MRS} = \left(\frac{c(t)}{c(s)}\right)^{\theta} \mathrm{e}^{-\rho(s-t)}.$$

and the intertemporal elasticity of substitution is

$$\eta_{(c(t)/c(s))/\text{MRS}} = \frac{\partial \ln\left(\frac{c(t)}{c(s)}\right)}{\partial \ln(\text{MRS})} = \frac{1}{\theta}.$$

$\square$

The program of the household is

$$\begin{aligned}
&\max_{\{c(t),k(t)\}} U = \int_0^{+\infty} \mathrm{e}^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta}\, \mathrm{d}t \\
&\text{s. t.} \quad c + \dot{k} = f(k) - \delta k \\
&\qquad\quad k(0) = k_0.
\end{aligned}$$

The current–value Hamiltonian is a technique to transform a dynamic problem into a static one, where k is a state variable and λ is a costate variable (the shadow price of capital). The Hamiltonian is the instantaneous utility plus the change in capital stock valued according to λ

$$H(c, k, \lambda) = \frac{c^{1-\theta}}{1-\theta} + \lambda[f(k) - \delta k - c].$$

The first–order conditions are found from the Maximum Principle. Maximizing the Hamiltonian with respect to the control variable yields the optimal control

$$c(t)^{-\theta} = \lambda(t).$$

The first–order condition with respect to the costate variable yields the dynamic equation

$$\rho\lambda = [f'(k) - \delta]\lambda + \dot{\lambda}.$$

From the first–order conditions

$$\frac{\dot{\lambda}}{\lambda} = \rho + \delta - f'(k),$$

$$-\theta \ln c = \ln \lambda.$$

By taking a time derivative on both sides

$$-\theta \frac{\dot{c}}{c} = \frac{\dot{\lambda}}{\lambda},$$

and solving yields the Keynes-Ramsey rule for the growth of consumption

$$\boxed{\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho}{\theta}}$$

The Mangasarian sufficiency conditions are

- If $\lambda > 0$, then the Hamiltonian $H(c, k, \lambda)$ is strictly jointly concave in (c, k).
- The transversality condition holds

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda(t)k(t) = 0.$$

Note that this is similar to a complementary slackness condition.

Definition: Equilibrium of the Ramsey Model
An equilibrium of the Ramsey model is a pair of functions $\big(c(t), k(t)\big)$ satisfying

$$\frac{\dot c}{c} = \frac{f'(k) - \delta - \rho}{\theta} \tag{1}$$

$$\dot k = f(k) - \delta k - c \tag{2}$$

$$\lim_{s\to+\infty} \mathrm{e}^{-\rho s}\frac{k(s)}{[c(s)]^{\theta}} = 0 \tag{3}$$

$$k(0) = k_0 \tag{4}$$

1.7.2. The Steady–State of the Ramsey Model

Next, a steady–state is a pair (k, c) such that $\dot k = 0$ and $\dot c = 0$. This respectively implies

$$f'(k) = \rho + \delta$$
$$f(k) - \delta k = c$$

Notice that the steady–state capital stock decreases with δ and ρ, and the capital stock increases if the production technology, $f(k)$, becomes more efficient. Also notice that the agents' willingness to smooth consumption across time does not influence the steady–state capital stock.

Notice that steady–state investment is $f(k) - c = \delta k$. Thus, the saving rate at the steady–state is

$$s = \frac{\delta k^*}{f(k^*)}.$$

The steady–state saving rate is increasing with k^*, and has a negative relationship with ρ (e.g. a decrease in ρ raises the saving rate).

Example: The Ramsey Model with Cobb-Douglas Production
Given a Cobb-Douglas production function

$$f(k) = Ak^{\alpha},$$

the steady–state, where $\dot k = 0$, occurs where

$$A\alpha k^{\alpha-1} = \rho + \delta.$$

Thus, steady–state capital stock is

$$k^* = \left(\frac{A\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}},$$

and the saving rate is

$$s = \frac{\delta k}{Ak^{\alpha}}$$

$$s = k^{1-\alpha}\frac{\delta}{A}$$
$$s = \left(\frac{A\alpha}{\rho+\delta}\right)^{\frac{1-\alpha}{1-\alpha}}\frac{\delta}{A}$$
$$s = \alpha\frac{\delta}{\rho+\delta}.$$

The capital stock that maximizes steady–state consumption is the golden–rule capital stock, k_{GR}, and satisfies

$$f'(k_{\text{GR}}) = \delta.$$

The capital stock in equilibrium satisfies $f'(k^*) = \delta + \rho$. Therefore, $k^* < k_{\text{GR}}$ and the economy is dynamically efficient.

1.7.3. The Dynamics of the Ramsey Model

Next, to study the properties of the dynamic system, linearize the transition equations in the neighborhood of the steady state

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{f''(k^*)c^*}{\theta} \\ -1 & \rho \end{pmatrix}}_{J} \begin{pmatrix} c-c^* \\ k-k^* \end{pmatrix}.$$

Notice that

$$\det J = \frac{f''(k^*)c^*}{\theta} < 0.$$

This implies that the steady–state is a saddle–point. The transversality condition is satisfied on the saddle–path. Indeed,

$$\lim_{s\to\infty} \lambda(s)k(s) = \lambda^* k^*,$$

and

$$\lim_{s\to\infty} \mathrm{e}^{-\rho s}\lambda(s)k(s) = 0.$$

From the strict concavity of the Hamiltonian and the Mangasarian sufficiency condition, the saddle path is the unique solution to the Ramsey problem.

Proof. Let $\mathbf{q}_1$ and $\mathbf{q}_2$ be the two eigenvectors and $\lambda_1 < 0$ and $\lambda_2 > 0$ be the two eigenvalues associated with the Jacobian matrix. It follows that

$$\begin{pmatrix} c-c^* \\ k-k^* \end{pmatrix} = C_1\mathbf{q}_1\mathrm{e}^{\lambda_1 t} + C_2\mathbf{q}_2\mathrm{e}^{\lambda_2 t}.$$

From the transversality condition

$$\lim_{t\to\infty}\begin{pmatrix} c-c^* \\ k-k^* \end{pmatrix} = 0,$$

which implies that $C_2 = 0$. Therefore,

$$c(t) = c^* + \mathrm{e}^{\lambda_1 t}[c(0)-c^*]$$
$$k(t) = k^* + \mathrm{e}^{\lambda_1 t}[k(0)-k^*].$$

□

The speed of adjustment to the steady–state is given by $|\lambda_1|$, where

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 - 4\frac{f''(k^*)c^*}{\theta}}}{2}.$$

The higher the elasticity of substitution, the faster capital accumulates. This is because people are more willing to accept low consumption early on in their life in exchange for higher consumption later.

A fall in ρ is the closest analogue to a rise in the saving rate in the Solow model. Suppose that the change is unexpected, that is, at some date households suddenly discover that they now discount utility at a lower rate. In the phase diagram, only the c-locus is affected which will lead to an increase in k^*. Note that k is a predetermined variable and cannot change discontinuously. In contrast, c can jump to a new value at any time.

1.8. Phase Diagrams

Phase diagrams are useful when conducting a qualitative analysis of a system of two differential equations, studying systems of nonlinear differential equations, or illustrating the different types of steady–state equilibria.

1.8.1. Construction of a Phase Diagram

Definition: A phase plane consists of a horizontal axis, y_1, and a vertical axis, y_2.

Definition: An isocline for y_i is the locus of points for which $\dot{y}_i = 0$

Definition: The isocline divides the phase plane into two isosectors. One where $\dot{y}_i$ is negative and the other where $\dot{y}_i$ is positive.

The two isoclines, for y_1 and y_2, intersect where both $\dot{y}_1$ and $\dot{y}_2$ equal zero. These are the steady–state points. From the two isoclines, the four quadrants can be deduced. In each quadrant, it is customary to draw arrows of motion to indicate how the system evolves.

- $\leftarrow$ and $\downarrow$ indicates that $\dot{y}_1 < 0$ and $\dot{y}_2 < 0$.
- $\leftarrow$ and $\uparrow$ indicates that $\dot{y}_1 < 0$ and $\dot{y}_2 > 0$.
- $\leftarrow$ and $\downarrow$ indicates that $\dot{y}_1 > 0$ and $\dot{y}_2 < 0$.
- $\rightarrow$ and $\uparrow$ indicates that $\dot{y}_1 > 0$ and $\dot{y}_2 > 0$.,

1.8.2. Vector Fields

The system of differential equations is interpreted as the equations of motion of a particle in the plane, with velocity vector $(\dot{y}_1, \dot{y}_2)$.

Definition: A vector field is a family of vectors where, for each point in the phase plane, you draw the velocity vector, $(\dot{y}_1, \dot{y}_2)$, with its tail at the point and pointing in the direction of the particle's motion.

Example: Phase Diagram with a Stable Node
Consider the following differential equation system

$$\dot{y}_1 = -2y_1 + 2$$
$$\dot{y}_2 = -3y_2 + 6$$

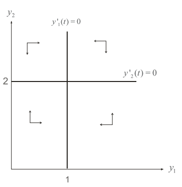
Written as $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{b}$, then

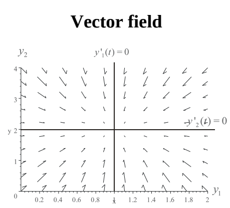
$$\mathbf{A} = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}.$$

The tr$\mathbf{A} = -5$, $\det \mathbf{A} = 6$, so $(\text{tr}\mathbf{A})^2 - 4\det\mathbf{A} = 25 - 24 = 1$. Thus, there is a stable node. The isoclines for y_1 and y_2 are as follows.

- If $\dot{y}_1 = 0$, then $y_1 = 2$.
- If $\dot{y}_1 > 0$, then $y_1 < 1$.
- If $\dot{y}_1 < 0$, then $y_1 > 1$.

- If $\dot{y}_2 = 0$, then $y_2 = 2$.
- If $\dot{y}_2 > 0$, then $y_2 < 2$.
- If $\dot{y}_2 < 0$, then $y_2 > 2$.





Example: Phase diagram with an Unstable Node
Consider the following system of differntial equations

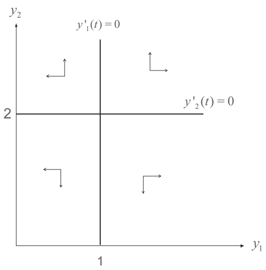
$$\dot{y}_1 = 2y_1 - 2$$
$$\dot{y}_2 = 3y_2 - 6$$

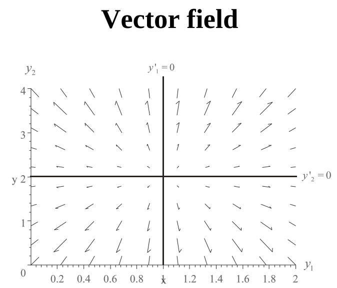
Written as $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{b}$, then

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

The tr$\mathbf{A} = 5$, $\det \mathbf{A} = 6$, so $(\mathrm{tr}\mathbf{A})^2 - 4\det \mathbf{A} = 25 - 24 = 1$. Thus, there is an unstable node. The isoclines for y_1 and y_2 are as follows.

- If $\dot{y}_1 = 0$, then $y_1 = 1$.
- If $\dot{y}_1 > 0$, then $y_1 > 1$.
- If $\dot{y}_1 < 0$, then $y_1 < 1$.
- If $\dot{y}_2 = 0$, then $y_2 = 2$.
- If $\dot{y}_2 > 0$, then $y_2 > 2$.
- If $\dot{y}_2 < 0$, then $y_2 < 2$.





Example: Phase Diagram with a Saddle Point
Consider the following differential equation system

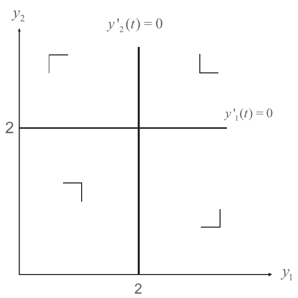
$$\dot{y}_1 = y_2 - 2$$
$$\dot{y}_2 = \frac{y_1}{4} - \frac{1}{2}$$

Written as $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{b}$, then

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{pmatrix}.$$

The tr$\mathbf{A} = 0$, $\det \mathbf{A} = -\frac{1}{4}$, so $(\mathrm{tr}\mathbf{A})^2 - 4\det \mathbf{A} = 0 - 4(-\frac{1}{4}) = 1$. Thus, there is a saddle point. The isoclines for y_1 and y_2 are as follows.

- If $\dot{y}_1 = 0$, then $y_2 = 2$.
- If $\dot{y}_1 > 0$, then $y_2 > 2$.
- If $\dot{y}_1 < 0$, then $y_2 < 2$.
- If $\dot{y}_2 = 0$, then $y_1 = 2$.
- If $\dot{y}_2 > 0$, then $y_1 > 2$.
- If $\dot{y}_2 < 0$, then $y_1 < 2$.



1.8.3. Finding a Saddle Path

In order to find a saddle path, first diagonalize the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}.$$

Then utilize the change of variables technique

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The system becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix}.$$

Notice that x_1 is unstable whereas x_2 is stable. The saddle path is such that x_1 is equal to its steady–state value so that it does not diverge

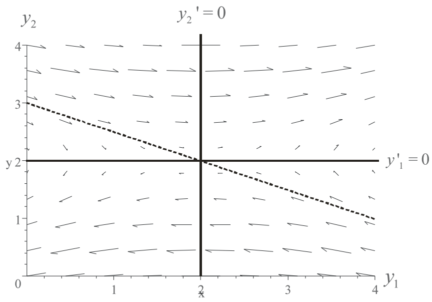
$$\dot{x}_1 = \frac{1}{2}x_1 - 3 = 0$$

$$x_1 = \overline{x}_1 = 6.$$

Using the fact that $x_1 = y_1 + 2y_2$, the equation of the saddle path is

$$y_1 + 2y_2 = 6.$$

Vector field



Example: A Phase Diagram with a Stable Focus
Consider the following differential equation system

$$\dot{y}_1 = -y_2 + 2$$
$$\dot{y}_2 = y_1 - y_2 + 1$$

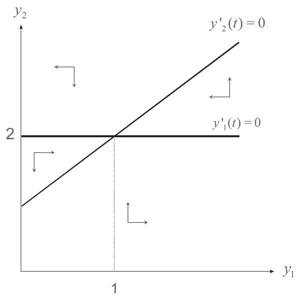
Written as $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{b}$, then

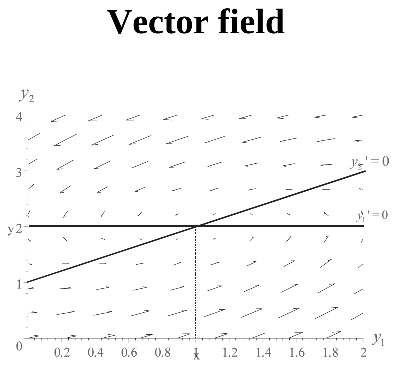
$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

The $\mathrm{tr}\mathbf{A} = -1$, $\det \mathbf{A} = 1$, so $(\mathrm{tr}\mathbf{A})^2 - 4 \det \mathbf{A} = 2 - 4(1) = -2 < 0$. Thus, there is a stable focus. The isoclines for y_1 and y_2 are as follows.

- If $\dot{y}_1 = 0$, then $y_2 = 2$.
- If $\dot{y}_1 > 0$, then $y_2 < 2$.
- If $\dot{y}_1 < 0$, then $y_2 > 2$.

- If $\dot{y}_2 = 0$, then $y_1 - y_2 = -1$.
- If $\dot{y}_2 > 0$, then $y_1 - y_2 > -1$.
- If $\dot{y}_2 < 0$, then $y_1 - y_2 < -1$.





There is a steady–state, where $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$, such that $y_1 = 1$ and $y_2 = 2$.

Example: The Dornbusch Model of Exchange–Rate Overshooting

A dynamic version of the Mundell–Fleming model helps analyze how exchange rates respond to a change in the money supply in an economy where the goods market does not clear instantaneously. The real demand for money is

$$m^D = -ar + b\overline{y}.$$

The equilibrium of the money market is

$$m - p = -ar + b\overline{y}.$$

There is perfect–foresight expectations of the depreciation of the national currency, $\dot{e}$, resulting in the interest rates parity

$$r = r^* + \dot{e},$$

where e is the exchange rate defined as the domestic price of foreign currency. From the two last equations

$$\dot{e} = -r + \frac{b\overline{y} - m + p}{a}.$$

There is sluggish adjustment of prices

$$\overline{p} = \alpha(y^D - \overline{y}),$$

where $\alpha > 0$ and aggregate demand is given by

$$y^D = u + v(e - p).$$

The differential equation for prices is then

$$\dot{p} = \alpha(u + ve - vp - \overline{y}).$$

The system of differential equations can be written in matrix form

$$\begin{pmatrix} \dot{p} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} -\alpha v & \alpha v \\ \frac{1}{\alpha} & 0 \end{pmatrix} \begin{pmatrix} p \\ e \end{pmatrix} + \begin{pmatrix} \alpha(u - \overline{y}) \\ \frac{b\overline{y} - m}{a} - r^* \end{pmatrix}.$$

Denote $\mathbf{A}$ the matrix of coefficients. Then

$$\det \mathbf{A} = -\frac{\alpha v}{a} < 0,$$

and all the roots are real valued and of opposite sign. Thus, the steady–state is a saddle–point. At the steady–state

$$\begin{pmatrix} \dot{p} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\alpha v & \alpha v \\ \frac{1}{\alpha} & 0 \end{pmatrix} \begin{pmatrix} p \\ e \end{pmatrix} = \begin{pmatrix} \alpha(u - \overline{y}) \\ \frac{b\overline{y}-m}{a} - r^* \end{pmatrix}$$

$$\begin{pmatrix} p \\ e \end{pmatrix} = \frac{a}{\alpha v} \begin{pmatrix} 0 & -\alpha v \\ -\frac{1}{a} & -\alpha v \end{pmatrix} \begin{pmatrix} \alpha(u - \overline{y}) \\ \frac{b\overline{y}-m}{a} - r^* \end{pmatrix}$$

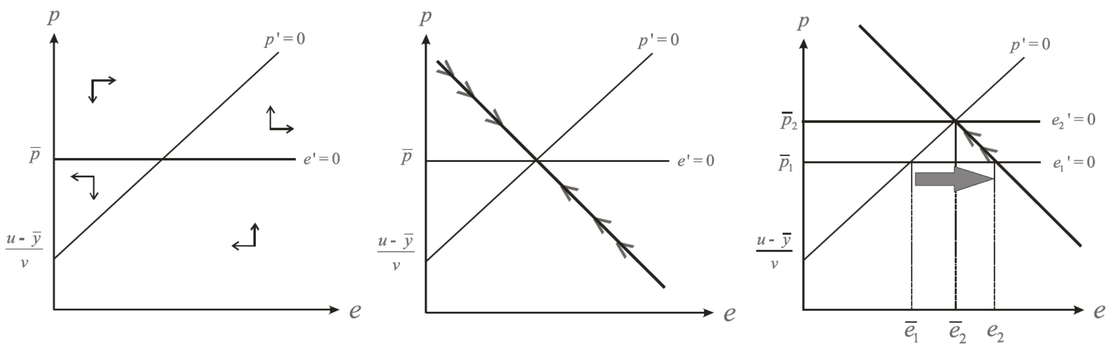
$$\begin{pmatrix} p \\ e \end{pmatrix} = \begin{pmatrix} -b\overline{y} + m + ar^* \\ -\frac{u-\overline{y}}{v} - b\overline{y} + m + ar^* \end{pmatrix}.$$

The p isocline in the phase diagram is

$$\dot{p} = 0$$
$$p = \frac{u - \overline{y}}{v} + e.$$

The e isocline in the phase diagram is

$$\dot{e} = 0$$
$$p = ar^* - b\overline{y} + m = \overline{p}.$$



The domestic price, p, changes sluggishly, because of the initial condition, p_0. The nominal exchange rate, e, can adjust instantly, because there is no initial value for e. To determine the trajectory of the economy, a condition imposed by the assumption of perfect foresight is that agents only anticipate trajectories that converge to the steady state

$$\lim_{t \to \infty} e(t) = \overline{e}.$$

Assume there is an initial point, steady–state $(\overline{p}_1, \overline{e}_1)$, and an increase in the money supply, m. Then, the p isocline is not affected, the e isocline moves upward, and in the long run, both p and e increase, while the real exchange rate rate remains unchanged. The increase in m triggers a jump of the exchange rate

$$\overline{e}_1 \to e_2.$$

Following the jump, the nominal exchange rate is larger than its new steady-state value, initially overshooting the new steady-state exchange rate

$$e_2 > \overline{e}_2.$$

1.9. The Neoclassical Growth Model in Continuous Time

The Ramsey problem is that an agent wish to maximize her lifetime utility subject to a technological constraint. This is equivalent to the program of a social planner. In the decentralized economy, assume that there are households who consume and supply labor services, and firms who rent capital and labor services and produce output. The objective is to discover if the decentralized equilibrium efficient.

1.9.1. Ramsey's Neoclassical Growth Model

First, time is continuous and infinite. there are a large number of identical firms with CRS production function

$$Y = F(K, L).$$

The firms hire workers and rent capital. The factor markets and output markets are competitive. The firms are owned by the households, where there are a large number, H, of identical households. Each household supplies 1 unit of labor at every point in time and rents its capital to the firms, where its initial capital is

$$\frac{K(0)}{H},$$

and capital depreciates at rate δ. There is a debt market in which households can borrow and lend. Loans and capital pay the same real rate of return, $r(t)$, and are thus perfect substitutes. The rental rate of capital is

$$p_k(t) = r(t) + \delta.$$

Both households and firms are price–takers and have perfect foresight expectations. This implies that current and future values of $r(t)$ and $w(t)$ are known

$$r(t) = p_k K, w(t) = wL.$$

The firm's behavior is equivalent to the static program

$$\max_{K,L} F(K, L) - p_k K - wL,$$

or equivalently in intensive form

$$\max_{k} k(k) - p_k k - w.$$

The first order conditions imply that

$$\begin{aligned} p_k &= f'(k), \\ w &= f(k) - p_k k. \end{aligned}$$

Note that

$$w = f(k) - f'(k)k.$$

Next, the household's utility function

$$U = \int_0^{+\infty} \mathrm{e}^{-\rho t} u(c(t))\,\mathrm{d}t,$$

takes the form of an instantaneous utility function

$$u(C) = \begin{cases} \frac{C^{1-\theta}}{1-\theta} & \text{if } \theta > 0, \\ \ln C & \text{if } \theta = 1, \end{cases},$$

where $\frac{1}{\theta}$ is the intertemporal elasticity of substitution. Let $a(t)$ denote the net value of the household's assets. All assets guarantee the same rate of return, r. The law of motion for the household's assets is

$$c + \dot{a} = w + ra$$

By the method of the integrating factor

$$\lim_{s \to +\infty} \mathrm{e}^{-R(s)} a(s) = a(0) + \int_0^{+\infty} \mathrm{e}^{-R(t)} [w(t) - c(t)]\,\mathrm{d}t,$$

where

$$R(t) = \int_0^t r(\tau)\,\mathrm{d}\tau.$$

There is a no–Ponzi game condition that is imposed

$$\lim_{s \to +\infty} \mathrm{e}^{-R(s)} a(s) \geq 0.$$

The present value of the household's asset holdings cannot be negative in the limit. So, someone cannot issue debt and roll it over forever

$$a(0) + \int_o^{+\infty} \mathrm{e}^{-R(t)} [w(t) - c(t)]\,\mathrm{d}t \geq 0.$$

The program of the household is then

$$\begin{aligned} \max_{\{c\}} U &= \int_0^{+\infty} \mathrm{e}^{-\rho t} \frac{C^{1-\theta}}{1-\theta}\,\mathrm{d}t \\ \text{s. t. } c + \dot{a} &= w + ra \\ \lim_{t \to +\infty} \mathrm{e}^{-R(t)} a(t) &\geq 0 \\ a(0) &= a_0. \end{aligned}$$

The current-value Hamiltonian is

The first–order condition corresponding to the optimal control is

$$c(t)^{-\theta} = \lambda,$$

where the costate variable, λ, is the shadow price of capital. The equation for the costate variable is

$$\rho\lambda = \lambda r + \dot{\lambda}.$$

If you take a time derivative of the first–order condition,

$$-\theta \dot{c} c^{-\theta-1} = \dot{\lambda},$$

and rearrange the costate equation, so that

$$\frac{\dot{\lambda}}{\lambda} = \rho - r.$$

Thus, optimal consumption growth is given by

$$\frac{\dot{c}}{c} = \frac{r-\rho}{\theta}.$$

The Mangasarian sufficient conditions must hold.

- The Hamiltonian $H(c, k, \lambda^*)$, is jointly concave in (c, k) where λ^* is generated by the maximum principle.
- The transversality condition holds

  $$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda^*(t)[a(t) - a^*(t)] \geq 0,$$

  where $a^*(t)$ is the candidate for a maximum and $a(t)$ is an alternative admissible trajectory.

The dynamic equation for λ implies

$$\lambda^* = \overline{\lambda}\mathrm{e}^{\rho t - R(t)}$$
$$\mathrm{e}^{-\rho t}\lambda^* = \overline{\lambda}\mathrm{e}^{-R(t)}.$$

The no–Ponzi game condition can then be rewritten as

$$\lim_{t\to+\infty} \mathrm{e}^{-\rho t}\lambda^*(t)a(t) \geq 0.$$

From the previous transversality condition, a sufficient condition for a maximum is that

$$\lim_{t\to+\infty} \mathrm{e}^{-\rho t}\lambda^*(t)a(t) = 0.$$

In equilibrium, The labor and the capital markets clear

$$L = H,$$
$$a = k.$$

Therefore, the law of motion for the household is

$$c + \dot{k} = f(k) - \delta k.$$

Definition: An equilibrium of the decentralized Ramsey model is a 4–tuple

$$\{c(t), k(t), r(t), w(t)\},$$

that satisfies

$$\begin{aligned}
\frac{\dot{c}}{c} &= \left(\frac{f'(k) - \delta - \rho}{\theta}\right), \\
\dot{k} &= f(k) - \delta k - c, \\
r(t) &= f'(k(t)) - \delta \\
w(t) &= f(k(t)) - f'(k(t))k(t) \\
\lim_{s\to+\infty} \mathrm{e}^{-\rho s}\frac{k(s)}{c(s)^{\theta}} &= 0 \\
k(0) &= k_0.
\end{aligned}$$

1.9.2. The Steady–State and Dynamics of the Ramsey Model

The decentralized Ramsey model is equivalent to the centralized Ramsey model. The models share the same phase diagram, steady–state and dynamics. In particular, the steady–state, where $k = k^*$, is

$$\begin{aligned} f'(k) &= \delta + \rho, \\ c &= f(k) - \delta k, \\ r^* &= \rho. \end{aligned}$$

There are the same first–order conditions as the ones of the competitive equilibrium. Thus, the decentralized equilibrium is Pareto efficient and satisfies the 1^{st} Welfare Theorem.

Note that from a an unanticipated fall in the discount rate, there is a new steady–state and a new saddle path. Initially, consumption falls, then capital converges asymptotically to its new steady–state value which is higher, resulting in higher long–run consumption.

1.9.3. The Neoclassical Growth Model with Government

Suppose that there is a government who buys output at rate $g(t)$ per unit of labor per unit time. Government purchases affect neither the utility of private agents nor the production technology. There is a balanced budget, financed by lump–sum taxes of amount $g(t)$. The law of motion for capital is

$$\dot{k} = f(k) - \delta k - c - g(t).$$

Notice that the k–locus is affected. The c-locus given by the Euler equation is unaffected by the presence of the government. The no–Ponzi game condition still holds

$$k(0) + \int_0^{+\infty} \mathrm{e}^{-R(t)} [w(t) - g(t) - c(t)] \, \mathrm{d}t \geq 0.$$

Example: A Non–Anticipated Increase in the Government's Spending
Suppose that there is a non–anticipated increase in the government's spending. The trajectory for government's expenditure is

$$g(t) = \begin{cases} g_0 & \text{for all } t < t_1, \\ g_1 > g_0 & \text{for all } t \geq t_1. \end{cases}$$

At $t = t_1$, the system jumps to its new steady–state, consumption falls by the exact amount of the increase in g, and capital stock is unchanged.

Example: An Anticipated Increase in the Government's Spending
Now, suppose that there is an anticipated increase in government's spending. At $t = t_0 < t_1$ the government announces the future increase in government's spending. The households want to smooth consumption, this implies that consumption starts to fall before taxes have increased. The capital stock increases and then decreases, and in the long run is unchanged.

Example: Ricardian Equivalence
Suppose that the path for government's spending, $g(t)$, is exogenous. The government's purchases can be financed through taxation or by issuing bonds. The initial debt of the

government per unit of labor is D_0. The budget constraint for the government is an identity

$$g(t) + rD = \dot{D} + T(t),$$

and there is a no–Ponzi game condition

$$\lim_{s \to +\infty} \mathrm{e}^{-R(s)} D(s) = 0.$$

The budget constraint of the government can be rewritten as

$$D_0 + \int_0^\infty \mathrm{e}^{-R(t)} g(t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-R(t)} T(t)\,\mathrm{d}t.$$

The budget constraint of households is

$$a(0) + \int_0^{+\infty} \mathrm{e}^{-R(t)} [w(t) - T(t)]\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-R(t)} c(t)\,\mathrm{d}t.$$

If you combine the two, then

$$a(0) - D_0 + \mathrm{e}^{-R(t)} [w(t) - g(t)]\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-R(t)} c(t)\,\mathrm{d}t.$$

The result is Ricardian equivalence, that is, the choice of $T(t)$ has no real consequences. Agents know that a reduction in taxes today has to be matched by an increase in taxes in the future. Following a decrease in taxes, households keep their consumption constant and raise their savings.

1.10. A Model of Endogenous Growth

In the Solow and Ramsey models, the rate of growth of the economy is exogenous, output per–capita grows at the rate of technological progress. The saving rate does influence transitional dynamics, but not long–run growth. To extend the Ramsey model to allow for endogenous growth, you can drop the assumption of diminishing returns to capital accumulation (Paul Romer, 1986). This is the AK approach of endogenous growth.

1.10.1. The Endogenous Growth Model

Assume that there is a household (or one agent) who is infinitely-lived. She chooses how much to consume and to save (invest) in order to maximize her lifetime utility. There is initial capital, $k(0)$, and no capital depreciation. There is a non–neoclassical production function. There is a linear production function

$$f(k) = Ak,$$

where the marginal product of capital, A, is not diminishing. Thus, the Inada conditions are violated

$$\lim_{k\to\infty} f'(k) = A \neq 0.$$

This is the key element that underlies endogenous growth.

The program of the household is

$$\begin{aligned} \max_{\{c\}} U &= \int_0^{+\infty} \mathrm{e}^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta}\,\mathrm{d}t \\ \text{s. t. } c + \dot{k} &= Ak \\ k(0) &= k_0, \end{aligned}$$

where it is assumed that $(1-\theta)A < \rho < A$. The current–value Hamiltonian is

$$(H, c, k, \lambda) = \frac{c(t)^{1-\theta}}{1-\theta} + \lambda[f(k) - c].$$

The first–order condition results in the optimal control

$$c(t)^{-\theta} = \lambda(t).$$

The equation for the costate variable is

$$\rho\lambda = A\lambda + \dot{\lambda}.$$

From the two previous equations it follows that

$$\begin{aligned} \dot{\lambda} &= -\theta \dot{c} c^{-\theta-1} \\ \frac{\dot{\lambda}}{\lambda} &= \rho - A. \end{aligned}$$

Thus,

$$\frac{\dot{c}}{c} = \frac{A-\rho}{\theta} > 0,$$

consumption growth does not depend on the stock of capital. Furthermore, from the first–order condition and the law of motion of capital, then

$$c + \dot{k} = Ak \geq 0,$$

and there is no steady–state with positive consumption.

1.10.2. The Balanced Growth Equilibrium

There is no steady state for k in the endogenous growth model. However, you can look for an equilibrium where k and c grow at constant rate and then the ratio (c/k) would be constant.

Definition: An equilibrium of the endogenous growth model is a pair of functions, $\{k(t), c(t)\}$, that satisfy

$$\frac{\dot{c}}{c} = \frac{A-\rho}{\theta} \qquad \text{and} \qquad \dot{k} = Ak - c,$$
$$\text{s. t.} \lim_{t\to\infty} \mathrm{e}^{-\rho t}\frac{k(t)}{c(t)^{\theta}} = 0 \qquad \text{and} \qquad k(0) = k_0.$$

First, define $z \equiv (c/k)$. To write a differential equation for z, you can take the log of z and differentiate with respect to time

$$\frac{\dot{z}}{z} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k}$$
$$\frac{\dot{z}}{z} = \frac{A-\rho}{\theta} - A + z$$
$$\frac{\dot{z}}{z} = \frac{A(1-\theta)-\rho}{\theta} + z.$$

Thus, the steady–state for $z = \frac{c}{k}$ is

$$z = \begin{cases} 0, \\ \frac{\rho - A(1-\theta)}{\theta}. \end{cases}$$

The transversality condition is

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda(t)k(t) = 0.$$

You then have

$$\lambda(t)k(t) = (\tfrac{k}{c})[c(t)]^{1-\theta}$$
$$\lambda(t)k(t) = (\tfrac{k}{c})[c(0)]^{1-\theta}\mathrm{e}^{\frac{A(1-\theta)-\rho}{\theta}t}.$$

At the steady-state value of $z = \frac{c}{k}$, then

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda(t)k(t) = \lim_{t\to\infty} (\tfrac{k}{c})[c(0)]^{1-\theta}\mathrm{e}^{\frac{A(1-\theta)-\rho}{\theta}t} = 0.$$

Under the balanced growth path, the ratio $\frac{c}{k}$ is constant and equal to

$$\left(\frac{c}{k}\right)^{*} = \frac{\rho - A(1-\theta)}{\theta},$$

and consumption, capital, and output are all growing at the same rate.

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{y}}{y} = \frac{A-\rho}{\theta}.$$

The ratio (c/k) is a jump variable, that is, it adjusts instantly to its steady–state value. The rates of growth of c, k, and y are constant and do not depend on any endogenous variables. There are no transitional dynamics.

The saving rate is given by

$$
\begin{aligned}
s &= 1 - \frac{c}{Ak} \\
s &= 1 - \frac{\rho - A(1-\theta)}{A\theta} \\
s &= \frac{A-\rho}{A\theta}.
\end{aligned}
$$

There is a relationship between the rate of growth and the saving rate

$$
As = \frac{\dot{y}}{y}.
$$

In the long–run, growth rate depends on the willingness to save and the productivity of capital. Lower values of ρ and θ imply a higher willingness to save, and this will result in a higher growth rate. An improvement in the level of technology will also lead to a higher growth rate. These are conclusions that are very different from those of the Ramsey model.

1.10.3. Endogenous Growth with Physical and Human Capital

A shortcoming of the previous model is that the share of capital in national income is equal to one. Consider the introduction of human capital, h, with a production function

$$
y = kf\left(\frac{h}{k}\right),
$$

that has CRS with respect to h and k, and $f(\cdot)$ is strictly concave. Let investment in physical capital be i. The program of the agent is

$$
\begin{aligned}
\max_{\{c\}} U &= \int_0^{+\infty} \mathrm{e}^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta}\,\mathrm{d}t \\
\text{s. t. } \dot{h} &= kf(\tfrac{h}{k}) - c - i \\
\dot{k} &= i \\
k(0) &= k_0.
\end{aligned}
$$

The Hamiltonian of the agent is

$$
H(k, h, c, i, \lambda_k, \lambda_h) = \frac{c(t)^{1-\theta}}{1-\theta} + \lambda_h \left[kf\left(\frac{h}{k}\right) - c - i \right] + \lambda_k i.
$$

The first–order conditions are

$$
\begin{aligned}
c^{-\theta} &= \lambda_h \\
\lambda_h &= \lambda_k.
\end{aligned}
$$

The shadow value of physical capital is

$$
\rho\lambda_k = \lambda_h[f(\tfrac{h}{k}) - f'(\tfrac{h}{k})\tfrac{h}{k}] + \dot{\lambda}_k,
$$

and the shadow value of human capital

$$
\rho\lambda_h = \lambda_h f'(\tfrac{h}{k}) + \dot{\lambda}_h,
$$

The condition $\lambda_h = \lambda_k$ gives

$$
f\left(\frac{h}{k}\right) - f'\left(\frac{h}{k}\right)\frac{h}{k} = f'\left(\frac{h}{k}\right).
$$

Thus, both types of capital have the same rate of return.

Let $\kappa = \frac{h}{k}$ denote the solution to

$$f(\kappa) - f'(\kappa)\kappa = f'(\kappa),$$

and let $A = f'(\kappa)$. The model reduces to the simple AK model described earlier. The rate of growth of consumption and output is given by

$$\frac{\dot{c}}{c} = \frac{\dot{y}}{y} = \frac{f'(\frac{h}{k}) - \rho}{\theta}.$$

1.10.4. Endogenous Growth with Knowledge Spillovers

Assume that each agent's knowledge is a public good, that is, knowledge of one agent spills over across the whole economy. The knowledge agents depends on the aggregate stock of capital through a learning–by–doing effect. The production for agent can then be written as

$$y_i = f(k_i, \overline{k}),$$

where $f(\cdot)$ is a CRS production function and $\overline{k}$ is aggregate capital stock. The program of the household

$$\begin{aligned} \max_{\{c\}} U &= \int_0^{+\infty} \mathrm{e}^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta}\, \mathrm{d}t \\ \text{s. t. } c + \dot{k} &= f(k, \overline{k}) \\ k(0) &= k_0. \end{aligned}$$

This is the standard Ramsey model with diminishing private returns of capital accumulation. The solution to the program is

$$\frac{\dot{c}}{c} = \frac{f_k(k, \overline{k}) - \rho}{\theta}.$$

All the agents choose the same capital stock, $k = \overline{k}$. The rate of growth of consumption and output in equilibrium is then

$$\frac{\dot{c}}{c} = \frac{\dot{y}}{y} = \frac{f_k(\overline{k}, \overline{k}) - \rho}{\theta}.$$

Example: Knowledge Spillovers with Cobb–Douglas Production
Given a Cobb–Douglas production function

$$f(k_i, \overline{k}) = A k_i^{\alpha} \overline{k}^{1-\alpha},$$

you can find the first order condition

$$f_k(k_i, \overline{k}) = A\alpha \left(\frac{\overline{k}}{k_i}\right)^{1-\alpha}.$$

Thus, the growth of consumption and output in equilibrium given Cobb–Douglas technology is

$$\frac{\dot{c}}{c} = \frac{\dot{y}}{y} = \frac{A\alpha - \rho}{\theta}.$$

Example: Knowledge Spillovers with a Social Planner
Consider a social planner who internalizes the spillovers of knowledge across the firms. The program of the social planner is

$$\begin{aligned} \max_{\{c\}} U &= \int_0^{+\infty} \mathrm{e}^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta} \, \mathrm{d}t \\ \text{s. t. } c + \dot{k} &= f(k, k) \\ k(0) &= k_0. \end{aligned}$$

The solution to the program is

$$\frac{\dot{c}}{c} = \frac{f_k(k, k) + f_{\overline{k}}(k, k) - \rho}{\theta}.$$

Thus, the rate of growth in the planned economy is greater than in the decentralized economy. Note that under the Cobb–Douglas specification, that the rate of growth should be

$$\frac{A - \rho}{\theta}.$$

The agents in the decentralized economy do not internalize the positive externality of capital accumulation from knowledge spillovers.

1.11. Dynamic Programming Applications

1.11.1. Optimal Unemployment

The labor market differs from other markets in that labor is not homogeneous. It takes time for a worker to find a job, and for a vacancy to find a worker. There are large flows of workers and jobs between activity and inactivity, and large and persistent stocks of vacancies and unemployed workers. This section looks at the trade sector of the labor market and its frictions. A modeling device, similar to the aggregate production function introduced by Pissarides (1985, 1990), takes the form

$$M = m(U, V),$$

where U is the number of unemployed workers, V is the number of vacancies, and M is the number of match creations (hires).

Properties:

- The matching function is increasing with respect to its two arguments

$$\begin{aligned} m_U &> 0, \\ m_V &> 0. \end{aligned}$$

- There cannot be match creation without agents to be matched on both sides of the market

$$m(0, V) = m(U, 0) = 0.$$

- The matching function is strictly concave with respect to each of its arguments

$$\begin{aligned} m_{UU} &< 0, \\ m_{VV} &< 0. \end{aligned}$$

- The matching function is homogenous of degree 1 with respect to U and V.

Consider two economies which differ only with respect to their size. Under CRS the two economies have the same unemployment rate. The matching technology can be rewritten as

$$M = m_U U + m_V V,$$

where

$$\begin{aligned} m_U &= \frac{\partial m}{\partial U}, \\ m_v &= \frac{\partial m}{\partial V}. \end{aligned}$$

The job finding rate of an unemployed worker is

$$p = \frac{m(U, V)}{U} = m(1, \theta),$$

where $\theta = V/U$ indicates market tightness.

Example: Cobb–Douglas Matching Technology
The Cobb–Douglas specification of the matching function is

$$m(U, V) = AU^{\alpha} V^{1-\alpha},$$

where the efficiency of the matching process $A > 0$, and $0 < \alpha < 1$. This specification is reasonably successful in empirical studies (Blanchard and Diamond, 1990).

Let the labor force be normalized to one. The measure of employed (employment rate) is $e(t)$ and the unemployment rate is $u(t) = 1 - e(t)$. There are $v(t)$ vacancies and a separation rate s. The law of motion for the employment rate is

$$\dot{e} = m(u, v) - se.$$

All agents are risk neutral and discount future utility at rate r. The income of an unemployed agent is b. The productivity of a match is y and there is a flow cost of opening a vacancy, γ. A benevolent planner who maximizes society's net output faces the optimization problem

$$\begin{aligned} \max_{\{e(t),v(t)\}} \int_o^\infty & \mathrm{e}^{-rt}[e(t)y + (1-e(t))b - v(t)\gamma]\,\mathrm{d}t \\ \text{s. t. } \dot{e}(t) &= m[1-e(t), v(t)] - se(t), \\ e(0) &= e_0. \end{aligned}$$

The planner is subject to the matching frictions as described by m. The current-value Hamiltonian is

$$H(e, v, \lambda) = e(y-b) + b - v\gamma + \lambda[m(1-e, v) - se],$$

where λ, the shadow value of a job, is the co–state variable. The optimal control is

$$\gamma = \lambda m_v(1-e, v).$$

The cost of opening a vacancy is equal to the shadow value of a job times the marginal contribution of a vacancy to the matching process. Equivalently,

$$\gamma = \overbrace{\frac{m(1-e,v)}{v}}^{\text{Vacancy Filling Rate}} \overbrace{\frac{m_v(1-e,v)v}{m(1-e,v)}}^{\text{Vacancy Share}} \lambda.$$

The equation for the co–state variable is

$$r\lambda = y - b - m_u(1-e, v)\lambda - s\lambda + \dot{\lambda},$$

and has the usual interpretation as an asset pricing equation. The term $b + m_u(1-e, v)\lambda)$ can be interpreted as the flow value of an unemployed worker. It can be rewritten as

$$b + \overbrace{\frac{m(1-e,v)}{u}}^{\text{Job Finding Rate}} \overbrace{\frac{m_u(1-e,v)u}{m(1-e,v)}}^{\text{Unemployed Share}} \overbrace{\lambda}^{\text{Match Value}}.$$

Example: Cobb–Douglas Matching Technology (Continued)
For simplicity, let

$$m(u, v) = A\sqrt{u}\sqrt{v}.$$

The optimal control implies that

$$\sqrt{\frac{v}{1-e}} = \frac{\lambda A}{2\gamma}.$$

The number of vacancies per unemployed, $\frac{v}{1-e}$ is called market tightness. The equation for the co–state becomes

$$r\lambda = y - b - \frac{\lambda A}{2}\sqrt{\frac{v}{1-e}} - s\lambda + \dot{\lambda}$$
$$r\lambda = y - b - \frac{1}{\gamma}\left(\frac{\lambda A}{2}\right)^2 - s\lambda + \dot{\lambda}.$$

An equilibrium can be reduced to a pair of functions, $e(t)$ and $\lambda(t)$, that satisfy the following system of differential equations

$$\dot{\lambda} = (r+s)\lambda + \frac{1}{\gamma}\left(\frac{\lambda A}{2}\right)^2 + b - y$$
$$\dot{e} = A^2\frac{\lambda}{2\gamma}(1-e) - se,$$

where the initial condition, $e(0) = e_0$, and the Mangasarian sufficiency condition hold from the the fact that the state variable is non negative

$$\lim_{t\to\infty} \mathrm{e}^{-rt}\lambda(t)e(t) = 0.$$

To reach the stationary solution, consider the solution such that

$$\dot{\lambda} = 0,$$
$$\dot{e} = 0.$$

Then, the shadow value of a job solves

$$\frac{1}{\gamma}\left(\frac{A}{2}\right)^2\lambda^2 + (r+s)\lambda - (y-b) = 0.$$

The positive root is

$$\lambda^* = \frac{2\gamma}{A^2}\left[\sqrt{(r+s)^2 + \frac{A^2}{\gamma}(y-b)} - (r+s)\right].$$

The employment rate is

$$e^* = \frac{A^2\lambda^*}{2\gamma s + A^2\lambda^*},$$

where $e^* < 1$. Thus, it is optimal to have some unemployment.

Next, you can linearize the first–order equations around their steady states

$$\begin{pmatrix} \dot{\lambda} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} (r+s) + \frac{2\lambda^*}{\gamma}(\frac{A}{2})^2 & 0 \\ A^2\frac{(1-e^*)}{2\gamma} & -\frac{A^2\lambda^*}{2\gamma} - s \end{pmatrix}\begin{pmatrix} \lambda - \lambda^* \\ e - e^* \end{pmatrix}.$$

The steady–state is a saddle point and the saddle path is such that $\lambda(t) = \lambda^*$ and

$$\dot{e} = e^* + (e_0 - e^*)\exp\left[-\left(A^2\frac{\lambda^*}{2\gamma} + s\right)t\right].$$

Graphically, the saddle path coincides with the λ–isocline, and in the space (e, λ) it is horizontal.

The model provides a rich set of comparative statics.

	r	s	y	b	A	γ
λ^*	$-$	$-$	$+$	$-$	$-$	$+$
e^*	$-$	$-$	$+$	$-$	$+$	$-$

A higher separation rate, s means a lower value of a match and higher unemployment. Higher productivity, y, means a higher value of a match and lower unemployment. Higher match efficiency, A, means lower value of a match and lower unemployment.

The speed at which employment converges to its steady–state value is

$$A^2 \frac{\lambda^*}{2\gamma} + s = \sqrt{(r+s)^2 + \frac{A^2(y-b)}{\gamma}} - r.$$

Notice that the transition to the steady–state is faster if y is high and b is low. If the cost to open vacancies, γ, is high (e.g., because if credit market frictions) then the speed of convergence is low.

If there is an unanticipated shock that raises workers' productivity, y, then the shadow value of a job, λ^*, increases. Note that λ jumps instantly to its new steady–state value, and that the same is true for market tightness. Employment and unemployment converge gradually to their steady–state values.

1.11.2. Search Unemployment

Consider a Walrasian frictionless market where workers can find a job instantly. The only decision of a worker is whether or not to participate in the market. Search theory helps describe the worker's optimal search strategy, that may be affected by the distribution of job offers, the job destruction rate, and search costs. The first model in the economics literature, by Stigler (1961, 1962), regards choosing the optimal size of a sample. Next, search models where applied to the labor market by McCall (1970) who suggested that searching is sequential. For a review, see Mortensen (1986).

Assume that time is discrete and represented by $t \in \mathbb{N}^*$. The lifetime utility of a worker is

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t U(c_t)\right],$$

where c_t is the consumption at time t, and $\beta \in (0,1)$. It is assumed that

$$\begin{aligned} U' &> 0 \\ U'' &< 0 \\ U(0) &= 0 \\ U'(0) &< \infty. \end{aligned}$$

The worker cannot borrow or lend, her consumption is equal to her earnings. The worker begins each period with a current wage offer, w. She then has two alternatives; she can can work at that wage, or she can search for a new wage offer. All wage offers lie in $[0, \overline{w}]$. Let f be the density of wages, w, on that interval and F the c.d.f. If the worker chooses to work during the current period, then with probability $1-s$ the same wage is available to her next period. With probability s she will lose her job at the beginning of the next period and begin the next period with a wage of 0.

For a recursive formulation of the problem, the state variable is the current wage, w, and the control variable is the search variable $y \in \{0,1\}$, where

$$y = \begin{cases} 0 & \text{if the worker searches,} \\ 1 & \text{if the workerworks at her current job.} \end{cases}$$

The value function is $v(w)$. If the worker chooses to work, $y = 1$, her expected present discounted value of utility is

$$v(y = 1) = U(w) + \beta[(1 - s)v(w) + sv(0)].$$

If the worker chooses to search instead, her expected utility is

$$v(y = 0) = 0 + \beta \int_0^{\overline{w}} v(x)f(x)\,\mathrm{d}x.$$

The value function solves the following Bellman equation

$$v(w) = \max_{y\in\{0,1\}} \left\{ U(w) + \beta[(1 - s)v(w) + sv(0)], 0 + \beta \int_0^{\overline{w}} v(x)f(x)\,\mathrm{d}x \right\}.$$

Note that $v(w)$ is bounded above by $U(\overline{w})/(1 - \beta)$ and below by $U(0) = 0$. Therefore, you can work with the space of bounded functions $B([0, \overline{w}])$ with the supremum metric. Also, since $U(w)$ is continuous, you can work with the subspace of continuous bounded functions $C([0, \overline{w}])$. You can then check that the Blackwell sufficient conditions are satisfied. First, the value function is monotonic. If $h \leq g$, then

$$\beta[(1 - s)h(w) + sh(0)] \leq \beta[(1 - s)g(w) + sg(0)]$$
$$\int_0^{\overline{w}} h(x)f(x)\,\mathrm{d}x \leq \int_0^{\overline{w}} g(x)f(x)\,\mathrm{d}x.$$

Therefore, $Th \leq Tg$. Second, the value function satisfies discounting

$$T(h + a) = Th + \beta a.$$

Theorem: The Banach Fixed–Point Theorem
The Banach fixed–point theorem states that given $C([0, \overline{w}])$ where the supremum metric is a complete metric space, then the mapping given by the Bellman equation is a contraction mapping.

Therefore, from Banach fixed–point theorem, there is a unique v that solves the Bellman equation. This is because of the properties of the value function. Note that $U(w)$ is increasing and you can work with a closed subspace of weakly increasing functions. Furthermore, $v(w)$ is weakly increasing. Let

$$A = \beta \int_0^{\overline{w}} v(x)f(x)\,\mathrm{d}x.$$

From the Bellman equation

$$v(0) = \max \left\{ \beta v(0), \beta \int_0^{\overline{w}} v(x)f(x)\,\mathrm{d}x \right\} = A$$
$$v(\overline{w}) > \beta \int_0^{\overline{w}} v(x)f(x)\,\mathrm{d}x = A.$$

The term $U(w) + \beta[(1 - s)v(w) + sv(0)]$ is strictly increasing in w. That is, at at $w = 0$ it is less than A, and at $w = \overline{w}$ it is greater than A. Therefore there is a unique reservation wage, $w^* \in (0, \overline{w})$, such that

$$v(w^*) = U(w^*) + \beta[(1 - s)v(w^*) + sv(0)] = A.$$

Solving for $U(w^*)$ yields

$$U(w^*) = (1 - \beta)A.$$

Quarter II

"My fees are not too high. Your wage scale may simply be too low."
– Jack Vance, *Showboat World*

The objective of this section is to introduce you to the theories and methods of dynamic, stochastic macroeconomics. The hallmarks of dynamic macroeconomic models are intertemporal decision making and stochastic equilibrium processes. In order to study these two elements, this section is structured to emphasize computational tools and dynamic economics. Quarter II begins with some of the basic tools of decision-making in uncertain environments. All modern macroeconomic theories build on the workhorse growth models that are covered in Quarter I. You can use the insights from those frameworks to build up, step by step, to the Dynamic Stochastic General Equilibrium (DSGE) Model. This section then concludes with applications of the DSGE model to business cycles, asset pricing, and monetary/fiscal policy.

2.1. Introduction to Stochastic Processes

"An economic model is a probability distribution over a sequence."
– Thomas J. Sargent

2.1.1. Stochastic Processes

Let x_t be a realization form a random variable X_t.

Definition: A stochastic process is a sequence $\{x_t\}$ defined on a probability space.

Definition: An auto–covariance function is defined as

$$\gamma_x(r,s) \equiv \operatorname{cov}(x_r, x_s) = \mathbb{E}(x_r - \mathbb{E}\, x_r)(x_s - \mathbb{E}\, x_s).$$

Definition: The stochastic process $\{x_t\}_{t=-\infty}^{t=\infty}$ is covariance stationary provided

- $\mathbb{E}\, x_t = m$ for all t,
- $\gamma_x(r,s) = \gamma_x(r+t, s+t)$ for all t.

For simpler notation, set $t = -s$ and it follows that

$$\gamma_x(r,s) = \gamma_x(r-s, 0).$$

Now, let h be the order of the autocovariance in

$$\gamma_x(h) = \gamma_x(h, 0).$$

- When $h = 0$, then $\gamma_x(0)$ is the variance.
- When $h = 1$, then $\gamma_x(1)$ is the 1^{st} order auto–covariance.

2.1.2. Time Series

It is helpful to introduced the autoregressive moving average (ARMA) representation.

Definition: An autoregressive moving average (ARMA) model is a linear representation of a stochastic process with constant coefficients.

An ARMA(p,q) representation of x_t is

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + z_t + \theta_1 z_{t-1} + \cdots + \theta_q z_{t-q},$$

or

$$\phi(L)x_t = \theta(L)z_t,$$

where L is a lag operator such that

$$\begin{aligned}\phi(L) &= 1 - \phi_1(L) - \dots - \phi_p L^p \\ \theta(L) &= 1 + \theta_1 L + \dots + \theta_q L^q.\end{aligned}$$

An AR(p) is simply an ARMA($p, 0$), and likewise a MA(q) is simply an ARMA($0, q$).

Claim: An AR(1) is stationary provided $|\phi_1| < 1$.

Proof.

$$x_t = \phi_1 x_{t-1} + z_t,$$

and can be rewritten as

$$x_t = \phi_1^{k+1}(x_{t-(k+1)}) + \sum_{j=0}^{k} \phi_1^j z_{t-j}.$$

Note that

$$\lim_{k\to 0} \phi_1^{k+1}(x_{t-(k+1)}) = 0.$$

It follows that

$$x_t = \sum_{j=0}^{\infty} \phi_1^j z_{t-j},$$

and

$$\mathbb{E}\, x_t x_{t-1} = \mathbb{E}(z_t + \phi_1 z_{t-1} + \phi_1^2 z_{t-2} + \dots)(z_{t-1} + \phi_1 z_{t-2} + \phi_1^2 z_{t-3} + \dots).$$

Suppose that $|\phi_1| > 1$. Then

$$x_t = \phi_1 x_{t-1} + z_t,$$

can be rewritten as

$$x_t = \phi_1^{-1} x_{t+1} - \phi_1^{-1} z_{t+1}.$$

Continuing forward

$$x_t = -\sum_{j=1}^{\infty} \phi_1^j z_{t+j}.$$

$\square$

Definition: White noise is a process $\{z_t\} \overset{\text{i.i.d.}}{\sim} \mathcal{WN}(0, \sigma^2)$ if and only if

- $\mathbb{E}\, z_t = 0$ for all t,
- $\gamma_z(h) = \begin{cases} \sigma^2 & \text{if } h = 0, \\ 0 & \text{otherwise.} \end{cases}$

2.1.3. Markov Processes

Definition: Markov Property

$$\Pr(X_{t+1}|X_t, X_{t-1}, \dots) = \Pr(X_{t+1}|X_t).$$

A Markov process is a continuous state process if the Markov property holds.

A Markov chain is a finite state process if the Markov property holds.

Assume that X_t follows a Markov chain with a transition matrix P, where

$$P_{ij} = \Pr(x_{t+1} = e_j | x_t = e_i),$$

and e_i is a selector vector.

2.1.4. Rational Expectations

An expectational difference equation is a process

$$y_t = \alpha + \beta\,\mathbb{E}_t\, y_{t+1} + \gamma z_t, \tag{1}$$

where $z_t = ez_{t-1} + \varepsilon_t$ and $-1 < e < 1$.

Definition: A rational expectations equilibrium is a sequence $\{y_t\}$ that is a non–explosive solution to the expectational difference equation (1).

Finding solutions may lead to multiple equilibria, in which case refinement is needed. One such refinement is the Minimal State Variable (MSV) method. The MSV method finds a linear process for y_t with constant coefficients that depends on a minimal number of variables. The guess and verify method is as follows.

- First, guess
$$y_t = a + bz_t,$$
take future expectations
$$\mathbb{E}_t\, y_{t+1} = a + b\,\mathbb{E}_t\, z_{t+1},$$
substitute for $\mathbb{E}_t\, z_{t+1}$
$$\mathbb{E}_t\, y_{t+1} = a + bez_t,$$
then substitute into the difference equation (1)
$$y_t = \alpha + \beta a + \beta bez_t + \gamma z_t.$$
According to the guess, it is implied that
$$\begin{aligned} a &= \alpha + \beta a, \\ b &= \beta be + \gamma. \end{aligned}$$
Solving yields
$$\begin{aligned} \bar{a} &= \frac{\alpha}{1-\beta}, \\ \bar{b} &= \frac{\gamma}{1-\beta e}. \end{aligned}$$
The MSV is
$$y_t = \frac{\alpha}{1-\beta} + \frac{\gamma}{1-\beta e} z_t.$$
- Next, verify
$$\mathbb{E}_t\, y_{t+1} = \frac{\alpha}{1-\beta} + \frac{\gamma}{1-\beta e} ez_t,$$
and
$$y_t = \alpha + \frac{\alpha\beta}{1-\beta} + \frac{\gamma\beta}{1-\beta e} ez_t + \gamma z_t.$$
This is the rational expectations equilibrium.

Suppose that you wish to find all the solutions to a rational expectations model. First, note that the forecast error is

$$\eta_{t+1} = y_{t+1} - \mathbb{E}_t\, y_{t+1}.$$

There are rational expectations if

$$\mathbb{E}_t\, \eta_{t+1} = 0.$$

Note that

$$\mathbb{E}_t\, y_{t+1} = y_{t+1} - \eta_{t+1}.$$

Substituting into the expectational difference equation yields

$$\begin{aligned} y_t &= \alpha + \beta\, \mathbb{E}_t\, y_{t+1} + \gamma z_t \\ y_t &= \alpha + \beta(y_{t+1} - \eta_{t+1}) + \gamma z_t. \end{aligned}$$

If you assume that $\alpha = 0$ and $\gamma = 0$, then it follows that

$$y_{t+1} = \beta^{-1} y_t + \eta_{t+1}.$$

If $|\beta| < 1$, then the only solution is $\beta = 0$, $y_t = 0$, $\eta_t = 0$, and the model is determinate. If $|\beta| > 1$, then there is a continuum of solutions with y_0, sequence of η_t, $E\eta_{t+1} = 0$, and the model is indeterminate.

Suppose the guessed belief is

$$y_t = \beta^{-1} y_{t-1} + \eta_t.$$

This implies that future expectations are given by

$$\begin{aligned} \mathbb{E}_t\, y_{t+1} &= \beta^{-1} y_t + \mathbb{E}_t\, \eta_{t+1} \\ \mathbb{E}_t\, y_{t+1} &= \beta^{-1}(\beta^{-1} y_{t-1} + \eta_t) + 0 \\ \mathbb{E}_t\, y_{t+1} &= \beta^{-2} y_{t-1} + \beta^{-1}\eta_t. \end{aligned}$$

You can now substitute into the expectational difference equation

$$\begin{aligned} y_t &= \beta\, \mathbb{E}_t\, y_{t+1} \\ y_t &= \beta(\beta^{-2} y_{t-1} + \beta^{-1}\eta_t). \end{aligned}$$

This verifies that the rational expectations equilibrium is

$$y_t = \beta^{-1} y_{t-1} + \eta_t.$$

Example: Rational Expectations with an AR(1) Exogenous Process
Consider an expectational difference equation

$$y_t = \alpha + \beta\,\mathbb{E}_t\,y_{t+1} + \gamma z_t,$$

where the exogenous variable follows an AR(1) process

$$z_t = \rho z_{t-1} + \varepsilon_t.$$

Under rational expectations,

$$\mathbb{E}_t\,y_{t+1} = y_{t+1} - \eta_{t+1}.$$

If you set $\alpha = 0$, then

$$y_t = \beta y_{t-1} - \beta\eta_{t+1} + \gamma z_t.$$

You can then write the system in matrix notation as

$$.$$

2.2. Real Business Cycle Models

Begin with the social planner's problem

$$\max_{\{c_t,x_t,l_t,n_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\begin{aligned} \text{s. t. } c_t + x_t &= A_t F(k_t, n_t) \\ k_{t+1} &= (1-\delta)k_t + x_t \\ n_t + l_t &= 1. \end{aligned}$$

Assume that $F(k_t, n_t)$ has constant returns to scale.

The intertemporal condition (Euler equation) is

$$u'(c_t, 1-n_t) = \beta\, \mathbb{E}_t\, u'(c_{t+1}, 1-n_{t+1})[\underbrace{A_{t+1}F_k(k_{t+1}, n_{t+1}) + (1-\delta)}_{[A_{t+1}F_k(k_{t+1},n_{t+1})+(1-\delta)]\equiv 1+r_{t+1}}]. \tag{1}$$

The intratemporal condition (labor/leisure choice) is then found

$$-u_l(c_t, 1-n_t) + \lambda \underbrace{A_t F_n(k_t, n_t)}_{A_tF_n(k_t,n_t)\equiv \text{MPL}=w_t} = 0$$

$$\lambda = u_c(c, 1-n)$$

$$\frac{u_l(c_t, 1-n_t)}{u_c(c_t, 1-n_t)} = A_t F_n(k_t, n_t) \equiv w_t. \tag{2}$$

Also, note that the resource constraint is

$$c_t + x_t = A_t F(k_t, n_t) + (1-\delta)k_t. \tag{3}$$

The solution is the sequence $\{c_t, x_t, n_t\}_{t=0}^{t=\infty}$ that solves the intertemporal condition (1), the intratemproal condition (2), and satisfies the resource constraint (3). A standard technique is to log–linearize the equations, (1), (2), and (3), around their steady–state.

Notice that

$$\mathbb{E}_t\, u_c(c_{t+1}, 1-n_{t+1})(1+r_{t+1}) = \mathbb{E}_t\, u_c(c_{t+1}, 1-n_{t+1})\, \mathbb{E}_t(1+r_{t+1}) + \text{cov}_t\big(u_c(c_{t+1}, 1-n_{t+1}), (1+r_{t+1})\big).$$

- If $\text{cov}_t\big(u_c, (1+r_{t+1})\big) < 0$, then when r_{t+1} is high, consumption is also high.
- If $\text{cov}_t\big(u_c, (1+r_{t+1})\big) > 0$, then when r_{t_1} is high, consumption is low.

2.2.1. Solving Real Business Cycle Models with Bellman Equations

The social planner's problem can be setup using a Bellman equation and solved using various methods. Take note that the state variables are k and A, and that the control variables are c, n, and k'. The value function is

$$\begin{aligned} V(k, A) = \max_{\{c,n,k'\}} u(c, 1-n) + \beta\, \mathbb{E}_{A'|A}\, V(k', A') \\ \text{s. t. } c + k' = AF(k, n,) + (1-\delta)k. \end{aligned}$$

Using the Envelope Theorem

$$V_k(k, A) = u_c(c, 1-n)\frac{\mathrm{d}c}{\mathrm{d}k} = u_c(c, 1-n)[AF_k(k, n) + (1-\delta)].$$

The resulting policy functions are

$$\begin{aligned} c_t &= c(k_t, A_t) \\ n_t &= n(k_t, A_t) \\ k_{t+1} &= k(k_t, A_t). \end{aligned}$$

The corresponding Lagrangian is

$$\mathscr{L} = u(c, 1-n) + \beta\,\mathbb{E}_{A'|A}\,V(k', A') - \lambda(c + k' - AF(k, n) - (1-\delta)k).$$

The first–order conditions are

$$\begin{aligned} u_c(c, 1-n) - \lambda &= 0 \\ -u_l(c, 1-n) + \lambda AF_n(k, n) &= 0 \\ -\lambda + \beta\,\mathbb{E}_{A'|A}\,V_k(k', A') &= 0. \end{aligned}$$

The Euler (intertemporal) equation is

$$u_c(c, 1-n) = \beta\,\mathbb{E}_{A'|A}\,u_c(c', 1-n')[A'F_k(k', n') + (1-\delta)].$$

"The opportunity cost of another unit of capital."

"The return to savings (MPK + undeprecitated capital) valued at next period's MU."

2.2.2. Real Business Cycle Models with Productivity Shocks

If A_t increases temporarily there are important effects.

Definition: A wealth effect occurs when $A_tF(k_t, n_t)$ increases and causes c_t to increase[1].

Definition: An income effect occurs when the MPL increases and causes an increase in w_t, $\frac{u_l}{u_c}$, and l_t (a decrease in n_t).

Definition: A substitution effect occurs when the MPL increases and causes an increase in w_t, that in turn causes n_t to increase[2].

Definition: An interest rate effect occurs when the MPK increases and causes an increase in r_{t+1}, that in turn causes more capital to be accumulated, which will lead to an increase in future output and consumption.

If A_t increases for one period, then short–run wages are greater than long–run wages and labor, n_t, will increase.

If A_t increases permanently, then long–run wages, w, increase and labor, n_t, is substituted for leisure, l_t, in the short–run, that is, n_t will decrease.

[1] With consumption smoothing, then $\Delta c_t < \Delta y_t$.

[2] Empirically, the substitution effect usually dominates the income effect in most applications.

2.2.3. Calibration of Real Business Cycle Models

Two main goals for calibration of the RBC model are choosing the parameters for the utility and production functions. There are concerns that parameters of the utility or production function may not satisfy the balanced growth path restrictions. Typically, parameters are chosen from microeconomic studies. Estimating the production function is often of key importance.

Example: Estimation of a Cobb-Douglas Production Function
Given Cobb-Douglas technology

$$Y_t = A_t k_t^{\alpha} n_t^{1-\alpha},$$

a researcher could attempt to estimate

$$\log\left(\frac{y_t}{n_t}\right) = \alpha \log\left(\frac{k_t}{n_t}\right) + \log(A_t).$$

Definition: The Solow residual is $\log(A_t)$.

The Solow residual can be estimated by an AR(1) process

$$\log(A_t) = \rho \log(A_{t-1}) + \varepsilon_t.$$

Reading: See Prescott “Theory Ahead of Business–Cycle Measurement”.

Reading: Robert King and Sergio Rebelo (1999). “Resuscitating Real Business Cycles,” *Handbook of Macroeconomics*.

2.3. Asset Pricing and Financial Markets

The macroeconomic approach is based on consumption and savings models. Here, you will be introduced to complete markets of financial assets.

2.3.1. Complete Markets

In a complete market, there are prices for assets that pay off in a given state (i.e. a price for 1 unit of consumption at time t in state s_t). There is an introduction of a redundant asset, a 'risk–free bond'. Households make consumption decisions for the future and the future is uncertain (i.e. there are individual and aggregate shocks). Financial markets allow agents to smooth consumption across dates and states. This is a core component of modern macroeconomic models. Note that you cannot use Bellman equations to solve complete market models, because you must calculate summations over dates and states. The primary results of complete markets are that the distribution of wealth does not affect the allocation and the agents can fully insure against idiosyncratic risks.

2.3.2. The Lucas Asset Pricing Model

Now, consider the Lucas Asset Pricing Model where there is a representative household. There is one durable good–exogenous endowments of the 'Lucas tree', N_t. Trees yield a stochastic flow of nondurable goods–dividends or 'fruit', y_t. Consuming fruit yields utility, $u(c_t)$. Furthermore, households can purchase claims to trees, N_t, that are traded at price P_t. Household's can also buy or sell risk–free bonds with a face value of 1 unit of consumption. Let L_t be the holdings of bonds with bond price

$$R_t^{-1} = \frac{1}{1+r}.$$

The household's optimization problem is

$$\max_{\{C_t,N_t,L_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t)$$

$$\text{s. t. } C_t + P_t N_t + R_t^{-1} L_t = (P_t + y_t) N_{t-1} + L_{t-1}.$$

The first–order conditions are

$$\begin{aligned} u'(C_t) - \lambda_t &= 0 \\ -P_t \lambda_t + \mathbb{E}_t (P_{t+1} + y_{t+1}) \lambda_{t+1} &= 0 \\ -R_t^{-1} \lambda_t + \mathbb{E}_t \, \lambda_{t+1} &= 0. \end{aligned}$$

The Euler equations are

$$\begin{aligned} R_t^{-1} &= \beta \, \mathbb{E}_t \left[\frac{u'(C_{t+1})}{u'(C_t)} \right] \\ P_t &= \beta \, \mathbb{E}_t \left[\frac{u'(C_{t+1})}{u'(C_t)} \right] (P_{t+1} + y_{t+1}). \end{aligned}$$

The result is complete market prices, $\{P_t, R_t^{-1}\}$, for the durable good and the bond. There is an equilibrium if and only if the following conditions hold.

- There is a sequence $\{C_t, N_t, L_t\}$ that solves the household's optimization problem given complete market prices $\{P_t, R_t^{-1}\}$.

- All markets clear,

$$C_t = y_t$$
$$N_t = N$$
$$L_t = 0,$$

including the bond market.

2.3.3. The Martingale Theory of Stock–Prices

Definition: A martingale is a process x_t such that $\mathbb{E}[x_t + 1] = x_t$.

Note that in the Lucas Asset Pricing Model

$$P_t = \beta\,\mathbb{E}_t\left[\frac{u'(y_{t+1})}{u'(y_t)}\right]\mathbb{E}_t[P_{t+1} + y_{t+1}] + \beta\text{cov}\left[\frac{u'(y_{t+1})}{u'(y_t)}, P_{t+1} + y_{t+1}\right].$$

If

- the ratio $\left[\frac{u'(y_{t+1})}{u'(y_t)}\right]$ is constant
- and $\text{cov}\left[\frac{u'(y_{t+1})}{u'(y_t)}, P_{t+1} + y_{t+1}\right] = 0$,

then the household is risk neutral and

$$P_t = \beta\,\mathbb{E}_t[P_{t+1} + y_{t+1}].$$

If this is correct, then the future price for discounted dividends is the best forecast of future prices, P_t. Alternatively, P_t contains all useful information about future payoffs. This result is the Efficient Market Hypothesis.

2.3.4. Term Structure of Interest Rates

Now, consider a term structure of interest rates in the Lucas Asset Pricing Model. For simplicity, assume that there are 1 and 2 period bonds, the analysis can be readily extended with the addition of longer period bonds. There are two strategies for the household.

- Strategy 1: buy $\frac{1}{P_{1,t}}$ and receive payoff $\frac{1}{P_{1,t}}$.
- Strategy 2: buy $\frac{1}{P_{2,t}}$ and receive payoff $\frac{1}{P_{2,t}}P_{1,t+1}$.

It is assumed that there is no arbitrage

$$\frac{1}{P_{1,t}} = \frac{1}{P_{2,t}}P_{1,t+1}.$$

It follows that

$$P_{2,t} = P_{1,t}P_{1,t+1}$$
$$R_{2,t}^{-1} = R_{1,t}^{-1}R_{1,t+1}^{-1}$$
$$R_{2,t} = R_{1,t}\,\mathbb{E}_t[R_{1,t+1}].$$

The household's optimization problem is

$$\max_{\{C_t,N_t,L_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t)$$

$$\text{s. t. } C_t + P_tN_t + R_{1,t}^{-1}L_{1,t} + R_{2,t}^{-1}L_{2,t} = (P_t + y_t)N_{t-1} + L_{1,t-1} + R_{1,t}^{-1}L_{2,t-1}.$$

The first order conditions are

$$
\begin{aligned}
u'(C_t) - \lambda_t &= 0 \\
-P_t\lambda_t + \beta\,\mathbb{E}_t\,\lambda_{t+1}(P_{t+1} + y_{t+1}) &= 0 \\
-R_{1,t}^{-1}\lambda_t + \beta\,\mathbb{E}_t\,\lambda_{t+1} &= 0 \\
-R_{2,t}^{-1}\lambda_t + \beta\,\mathbb{E}_t\,R_{1,t+1}^{-1}\lambda_{t+1} &= 0.
\end{aligned}
$$

The Euler equations is

$$R_{2,t}^{-1} = \beta\,\mathbb{E}_t\,R_{1,t}^{-1}\left[\frac{u'(y_{t+1})}{u'(y_t)}\right]$$

or

$$R_{2,t}^{-1} = \beta\,\mathbb{E}_t\,R_{1,t}^{-1}\,\mathbb{E}_t\left[\frac{u'(y_{t+1})}{u'(y_t)}\right] + \beta\mathrm{cov}_t\left[R_{1,t+1}^{-1}, \frac{u'(y_{t+1})}{u'(y_t)}\right].$$

Note that the price of a 1–period bond is

$$R_{1,t}^{-1} = \beta\,\mathbb{E}_t\left[\frac{u'(y_{t+1})}{u'(y_t)}\right].$$

Thus the price of a 2–period bond is,

$$R_{2,t}^{-1} = R_{1,t}^{-1}\,\underset{\substack{\uparrow \\ \text{Expectations hypothesis.}}}{\mathbb{E}_t\,R_{1,t+1}^{-1}} + \underset{\substack{\uparrow \\ \text{Risk Premium.}}}{\beta\mathrm{cov}_t\left[R_{1,t+1}^{-1}, \frac{u'(y_{t+1})}{u'(y_t)}\right]}$$

If the $\mathrm{cov}_t < 0$, then the price of a 1–period bond is positively correlated with consumption growth. This pushes price down on a 2–year bond and increases the return, because of the bond–price risk.

2.3.5. The Equity Premium Puzzle

In the United States, as a rough estimate, the average rate of return on a bond is

$$r_{\mathrm{US},t+1} \approx 6\%.$$

Let

$$1 + r_{i,t+1} \equiv R_{i,t+1},$$

with Euler equations

$$1 = \beta\,\mathbb{E}_t\left[(1 + r_{i,t+1})\frac{u'(c_{t+1})}{u'(c_t)}\right],$$

for $i = s, b$. This may represent two different equities, such as a stock, s, and a bond, b. Assume that consumption growth is given by

$$\frac{c_{t+1}}{c_t} = \bar{c}_\Delta \exp\left(\varepsilon_{c,t+1} - \frac{\sigma_c^2}{2}\right),$$

where $\log(\varepsilon_c) \sim \mathcal{N}(0, \sigma_c^2)$. If you also assume C.R.R.A. utility, then

$$1 + r_{i,t+1} = (1 + \bar{r}_i)\exp\left(\varepsilon_{i,t+1} - \frac{\sigma_i^2}{2}\right),$$

where $\log(\varepsilon_i) \sim \mathcal{N}(0, \sigma_i^2)$ for $i = s, b$. Note that if x is log–normally distributed then $\mathbb{E}[x] = \exp(\mu + \frac{\sigma^2}{2})$. Therefore,

$$\mathbb{E}\left[\frac{c_{t+1}}{c_t}\right] = \overline{c}_\Delta \exp\left(0 + \frac{\sigma_c^2}{2} - \frac{\sigma_c^2}{2}\right) = \overline{c}_\Delta.$$

It follows that

$$1 = \beta(1+\overline{r}_i)\overline{c}_\Delta^{-\gamma}\, \mathbb{E}\left[\exp\left(\varepsilon_{i,t+1} - \frac{\sigma_i^2}{2} - \gamma\left(\varepsilon_{c,t+1} - \frac{\sigma_c^2}{2}\right)\right)\right].$$

Thus $\varepsilon_{i,t+1} - \gamma\varepsilon_{c,t+1}$ is log–normally distributed and

$$\begin{aligned}
\mathbb{E}[\varepsilon_{i,t+1} - \gamma\varepsilon_{c,t+1}] &= 0 \\
\mathrm{Var}[\varepsilon_{i,t+1} - \gamma\varepsilon_{c,t+1}] &= \sigma_i^2 - 2\gamma\mathrm{Cov}(\varepsilon_i, \varepsilon_c) + \gamma^2\sigma_c^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
1 &= \beta(1+\overline{r}_i)\overline{c}_\Delta^{-\gamma} \exp\left(\frac{\sigma_i^2}{2} - \gamma\mathrm{Cov}(\varepsilon_i, \varepsilon_c) + \frac{\gamma^2\sigma_c^2}{2} - \frac{\sigma_i^2}{2} + \gamma\frac{\sigma_c^2}{2}\right) \\
1 &= \beta(1+\overline{r}_i)\overline{c}_\Delta^{-\gamma} \exp\left((1+\gamma)\frac{\gamma\sigma_c^2}{2} - \gamma\mathrm{Cov}(\varepsilon_i, \varepsilon_c)\right).
\end{aligned}$$

If you take the natural logarithm of both sides, then

$$0 = \log\beta + \log(1+\overline{r}_i) - \gamma\log(\overline{c}_\Delta) + (1+\gamma)\frac{\gamma\sigma_c^2}{2} - \gamma\mathrm{Cov}(\varepsilon_i, \varepsilon_c).$$

You can now find that

$$\log(1+\overline{r}_s) - \log(1+\overline{r}_b) = \gamma\mathrm{Cov}(\varepsilon_s, \varepsilon_c) - \gamma\mathrm{Cov}(\varepsilon_b, \varepsilon_c).$$

So,

$$\underbrace{\overline{r}_s - \overline{r}_b}_{\text{Equity Premium}} \approx \underbrace{\gamma\mathrm{Cov}(\varepsilon_s, \varepsilon_c)}_{\text{Risk Premium}},$$

because empirically $\gamma\mathrm{Cov}(\varepsilon_b, \varepsilon_c) \approx 0$. Thus, the equity premium for holding stocks, s, versus bonds, b, increases with $\mathrm{Cov}(\varepsilon_s, \varepsilon_c)$.

2.3.6. Arrow–Debreu Securities

The framework is a representative agent structure. There are a finite number of people, $i = 1, 2, \ldots, I$. There is an idiosyncratic endowment process, $y_{i,t}(s_t)$, that depends on the state at a given time s_t.

- The state, s_t, is exogenous with finite values in set S (i.e. s_t is a Markov process).
- There is a history $S_t = [s_0, \ldots, s_t]$.
- $\mathbb{E}[S_t] = \pi_t(S_t)$.
- $\mathbb{E}[S_t|S_v] = \pi_t(S_t|S_v)$.

Furthermore, feasible allocations must satisfy

$$\sum_{i=1}^{I} c_{i,t}(S_t) = \sum_{i=1}^{I} y_{i,t}(S_t).$$

In the initial period, $t = 0$, the agents trade state–contingent consumption contracts. Each contract is a claim to 1 unit of consumption at time t contingent on being in state s_t. The price of a contract is

$$q_t^0(s_t).$$

The superscript denotes the date the contracts are traded and the subscript denotes the date when the goods are delivered. Agent i's expected life time utility is

$$U = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{i,t}(s_t)),$$

where $c_{i,t}(s_t)$ is the agent's consumption at time t at node s_t. The consumer's optimization problem is

$$\begin{aligned} \max_{\{c_{i,t}(s_t)\}} U &= \sum_{t=0}^{\infty} \sum_{s}^{S} \beta^t u(c_{i,t}(s_t))\pi_t(s_t) \\ \text{s. t.} \quad \sum_{t=0}^{\infty} \sum_{s}^{S} q_t^0(s_t) c_{i,t}(s_t) &= \sum_{t=0}^{\infty} \sum_{s}^{S} q_t^0(s_t) y_{i,t}(s_t), \end{aligned}$$

where π_t is the unconditional probability of history s_t. The Lagrangian for the consumer's problem is

$$\mathscr{L} = \sum_{t=0}^{\infty} \sum_{s}^{S} [\beta^t u(c_{i,t}(s_t))\pi_t(s_t) - \mu_i q_t^0(s_t)(c_{i,t}(s_t) - y_{i,t}(s_t))],$$

where μ_i is the multiplier on the consumer's lifetime budget constraint. The first–order conditions are

$$\frac{\partial \mathscr{L}}{\partial c_{i,t}(s_t)} = \beta^t u'(c_{i,t}(s_t))\pi_t(s_t) - \mu_i q_t^0(s_t) = 0,$$

for all t. If you combine the first–order conditions for agent i and j, then

$$\frac{u'(c_{i,t}(s_t))}{u'(c_{j,t}(s_t))} = \frac{\mu_i}{\mu_j}.$$

if you set $j = 1$ and solve for $c_{i,t}(s_t)$, then

$$c_{i,t}(s_t) = u'^{-1}\left(u'(c_{1,t}(s_t))\frac{\mu_i}{\mu_1}\right).$$

A feasible allocation must satisfy

$$\sum_{i=1}^{I} c_{i,t}(s_t) = \sum_{i=1}^{I} y_{i,t}(s_t).$$

Therefore,

$$\sum_{i=1}^{I} u'^{-1}\left(u'(c_{1,t}(s_t))\frac{\mu_i}{\mu_1}\right) = \sum_{i=1}^{I} y_{i,t}(s_t),$$

where $\sum_{i=1}^{I} y_{i,t}(s_t)$ is the aggregate endowment. You can now note that agent 1's consumption is a constant fraction of the aggregate endowment. Furthermore, agent i's consumption is a constant fraction of agent 1's consumption and thus a function of the aggregate endowment.

Now consider a framework with a social planner. The planner's optimization problem for is

$$\max_{\{c_t(s_t)\}} \mathbb{U} = \sum_{t=0}^{\infty} \sum_{s}^{S} \left[\sum_{i=1}^{I} \lambda_i \beta^t u(c_{i,t}(s_t))\pi_t(s_t) - \theta_t(s_t)(c_{i,t}(s_t) - y_{i,t}(s_t)) \right],$$

where λ_i is the Pareto weight on consumer i and $\theta_t(s_t)$ is the multiplier on the feasibility constraint. Note that $\theta_t(s_t)$ measures the shadow value of an additional unit of aggregate endowment. The first–order condition are

$$\frac{\partial \mathscr{L}}{\partial c_{i,t}(s_t)} = \lambda_i \beta^t u'(c_{i,t}(s_t))\pi_t(s_t) - \theta_t(s_t) = 0,$$

for all t and i. Note that the social planner's problem has the same first order conditions as the consumer's problem if

$$\lambda_i = \mu_i^{-1}$$
$$\theta_t(s_t) = q_t^0(s_t).$$

There are three major results.

1. The Pareto weight, λ_i, for agent i is inversely related to agent i's marginal utility of consumption. A high multiplier, μ_i, in the consumer's problem implies high marginal utility and low optimal consumption, thus a low Pareto weight, λ_i, for an equivalent allocation when there is a social planner.

2. The consumption contract prices, $q_t^0(s_t)$, equal the shadow value of aggregate endowment, $\theta_t(s_t)$. Thus, the economy satisfies the Second Welfare Theorem: out of all possible Pareto optimal outcomes, one can achieve any particular one by enacting a lump–sum wealth redistribution and then letting the market take over. Note that this requires a price at every node, s_t, and therefore a complete market.

3. Each agent i's consumption is a constant fraction of the aggregate endowment and the fraction is independent of any particular realized income history, $y_{i,t}(s_t)$. This result is known as risk–sharing, because although consumption remains a constant fraction of aggregate endowment, absolute consumption for all agents increases and decreases with the idiosyncratic aggregate endowment.

In conclusion, the Arrow-Debreu securities model is a complete market that provides full insurance against idiosyncratic risk.

Example: Arrow–Debreu Securities with No Aggregate Uncertainty
Consider an Arrow-Debreu securities market with two agents, $I = 2$, that have idiosyncratic endowments

$$\begin{aligned} y_{1,t} &= s_t \\ y_{2,t} &= 1 - s_t, \end{aligned}$$

where $s_t \in \{0, 1\}$. Note that the aggreate endowment is

$$\sum_{i=1}^{2} y_{i,t}(s_t) = 1.$$

You can deduce from the feasibility constraint that

$$\sum_{i=1}^{2} u'^{-1}\left(u'(c_{1,t}(s_t))\frac{\mu_i}{\mu_1}\right) = \sum_{i=1}^{2} y_{i,t}(s_t) = 1.$$

Thus, agent i's consumption is constant across dates and states

$$c_{i,t}(s_t) = \bar{c}_i.$$

The Euler equation for agent i is

$$q_t^0 = \frac{\beta^t u'(\bar{c}_i)\pi_t(s_t)}{\mu_i}.$$

The budget constraint of agent i is

$$\sum_{t=0}^{\infty}\sum_{s}^{S} q_t^0(s_t)[c_{i,t}(s_t) - y_{i,t}(s_t)] = 0.$$

It follows that

$$\begin{aligned} \sum_{t=0}^{\infty}\sum_{s}^{S} \frac{\beta^t u'(\bar{c}_i)\pi_t(s_t)}{\mu_i}[\bar{c}_i - y_{i,t}(s_t)] &= 0 \\ \frac{u'(\bar{c}_i)}{\mu_i}\left(\sum_{t=0}^{\infty}\sum_{s}^{S} \beta^t \pi_t(s_t)[\bar{c}_i - y_{i,t}(s_t)]\right) &= 0 \end{aligned}$$

$$\begin{aligned} \sum_{t=0}^{\infty}\beta^t \sum_{s}^{S} \pi_t(s_t)\bar{c}_i &= \sum_{t=0}^{\infty}\sum_{s}^{S} \beta^t y_{i,t}(s_t)\pi_t(s_t) \\ \frac{1}{1-\beta}(1)\bar{c}_i &= \sum_{t=0}^{\infty}\sum_{s}^{S} \beta^t y_{i,t}(s_t)\pi_t(s_t) \end{aligned}$$

because $\sum_s^S \pi_t(s_t) = 1$. Thus,

$$\bar{c}_i = (1-\beta)\underbrace{\sum_{t=0}^{\infty}\sum_{s}^{S} \beta^t y_{i,t}(s_t)\pi_t(s_t)}_{\text{Discounted expected lifetime income}}.$$

You can note that each agent smooths her consumption across dates and states, thus insured against the risk from the idiosyncratic income stream.

2.3.7. Economies with Incomplete Financial Markets

Consider a household that chooses consumption and savings with known endowments, $\{y_t\}$. The household may hold 1–period bonds, b_t, with a fixed rate of return, $R > 1$. Assume that $\beta R = 1$ or $R = \beta^{-1}$. Also assume that the household as an initial holding of bonds, b_0. Note that if $b_t < 0$, then the household is in debt. The household's problem is

$$\max_{\{c_t, b_{t+1}\}} U = \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s. t. } c_t + R^{-1} b_{t+1} \leq y_t + b_t.$$

Now consider possible debt limits that the household may face. A no–borrowing constraint states that $b_t \geq 0$. A natural debt limit is a maximum that can be borrowed and repaid

$$b_t \geq -\sum_{j=0}^{\infty} R^{-j} y_{t+j} \equiv \widetilde{b}.$$

A no–Ponzi–game condition states

$$\lim_{T \to \infty} R^{-T} b_{t+T} = 0.$$

Example: The Household's Problem with a No–Borrowing Constraint
The Lagrangian for the household's problem is

$$\mathscr{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) - \lambda_t (c_t + R^{-1} b_{t+1} - y_t - b_t) + \mu_t b_{t+1}.$$

The first–order conditions are

$$\frac{\partial \mathscr{L}}{\partial c_t} = u'(c_t) - \lambda_t = 0$$
$$\frac{\partial \mathscr{L}}{\partial c_t} = -R^{-1} \lambda_t + \beta \lambda_{t+1} + \mu_t = 0$$
$$\frac{\partial \mathscr{L}}{\partial \lambda_t} = c_t + R^{-1} b_{t+1} \leq y_t + b_t$$
$$\frac{\partial \mathscr{L}}{\partial \mu_t} = b_{t+1} \geq 0.$$

Note that

$$b_{t+1} > 0 \Leftrightarrow \mu_t = 0$$
$$b_{t+1} = 0 \Leftrightarrow \mu_t > 0,$$

and

$$c_t + R^{-1} b_{t+1} < y_t + b_t \Leftrightarrow \lambda_t = 0$$
$$c_t + R^{-1} b_{t+1} = y_t + b_t \Leftrightarrow \lambda_t > 0.$$

The Euler equation is

$$u'(c_t) = \beta R u'(c_{t+1}) + R \mu_t.$$

There are two potential cases.

(1) If $\mu_t = 0$, then $b_{t+1} > 0$ and there is perfect consumption smoothing

$$u'(c_t) = u'(c_{t+1})$$
$$c_t = c_{t+1} = \overline{c}.$$

(2) If $\mu_t > 0$, then $b_{t+1} = 0$ and the household is unable to shift consumption to the present as it is desirable to do

$$u'(c_t) > u'(c_{t+1})$$
$$c_t < c_{t+1}.$$

Thus, the household will set $c_t = y_t$ until

$$x_t \equiv \sum_{j=t}^{\infty} \beta^{j-t} y_j$$

is at its largest value, then $c_t = \overline{c}$ for the rest of the household's lifetime.

Example: A No–Borrowing Constraint with Idiosyncratic Endowment Stream Consider a household that receives income, y_i, that alternates between a high and low state, $y_i \in \{y_H, y_L\}$. Suppose that the household receives a random income stream

$$y_t = \{y_H, y_L, \dots\},$$

where $y_H > y_L$. Then discounted expected lifetime income is

$$x_0 = y_H + \beta y_L + \beta^2 y_H + \cdots = \frac{y_H + \beta y_L}{1 - \beta^2}$$
$$x_1 = y_L + \beta y_H + \beta^2 y_L + \cdots = \frac{y_L + \beta y_H}{1 - \beta^2}.$$

It follows that

$$\frac{\overline{c}}{1-\beta} = \frac{y_H + \beta y_L}{1 - \beta^2}$$
$$\overline{c} = \frac{y_H + \beta y_L}{1 + \beta}.$$

From the budget constraint and the assumption that $R = \beta^{-1}$ and $b_0 = 0$, you can find the household's holding of bonds

$$b_1 = R\left(y_H - b_0 - \frac{y_H + \beta y_L}{1 + \beta}\right)$$
$$b_1 = \beta^{-1}\left(\frac{\beta y_H - \beta y_L}{1 + \beta}\right)$$
$$b_1 = \frac{y_H - y_L}{1 + \beta} > 0$$

and

$$b_2 = \beta^{-1}\left(y_L + b_1 - \frac{y_H + \beta y_L}{1 + \beta}\right)$$
$$b_2 = \beta^{-1}\left(y_L + \frac{y_H - Y_L}{1 + \beta} - \frac{y_H + \beta y_L}{1 + \beta}\right)$$
$$b_2 = \beta^{-1}\left(\frac{y_L + \beta y_L + y_H - y_L - y_H - \beta y_L}{1 + \beta}\right)$$
$$b_2 = 0.$$

Suppose instead that the household receives a random income stream

$$y_t = \{y_L, y_H, \dots\}.$$

Then discounted expected lifetime income is

$$\begin{aligned}
x_0 &= y_L + \beta y_H + \beta^2 y_L + \dots = \frac{y_L + \beta y_H}{1-\beta^2} \\
x_1 &= y_H + \beta y_L + \beta^2 y_H + \dots = \frac{y_H + \beta y_L}{1-\beta^2}.
\end{aligned}$$

The household would like to borrow

$$b_1^* = \frac{y_L - y_H}{1+\beta} < 0,$$

but there is a no–borrowing constraint. Therefore,

$$\begin{aligned}
c_0 &= y_L \\
b_1 &= 0,
\end{aligned}$$

and the household can begin smoothing consumption starting in the subsequent period

$$\begin{aligned}
c_1 &= \overline{c} \\
b_2 &= \frac{y_H - y_L}{1+\beta} > 0.
\end{aligned}$$

The general result is cautionary savings where if the endowment stream is random and it is possible to realize an income that results in a binding no–borrowing constraint, then the household saves against those states. In equilibrium, there will be more savings and therefore a lower interest rate than when there are complete markets.

2.4. Applications of Dynamic Economics

For an introduction to applications of dynamic economics and quantitative methods used by modern macroeconomists, consider the general dynamic optimization problem

$$\max_{\{u_t\}_{t=0}^{\infty}} U = \sum_{t=0}^{\infty} \beta^t r(x_t, u_t)$$
$$\text{s. t. } x_{t+1} = g(x_t)$$
$$x(0) = x_0.$$

You can formulate the problem using a value function

$$V(x) = \max_u r(x,u) + \beta V(x')$$
$$\text{s. t. } x' = g(x),$$

where $r(x,u)$ is the return function, u is a control variable, x is a state variable, and $g(x)$ is a state–transition equation.

Theorem: Contraction Mapping Theorem

$$V_{j+1}(x) = \max_u r(x,u) + \beta V_j(x')$$
$$V_{j+1}(x) \equiv T^j V_j(x),$$

where T is an operator. The solution is a fixed point such that

$$V(x) = TV(x).$$

Finding the solution to a value function is known as iterating on the Bellman equation.

Theorem: Principle of Optimality
The solution to the sequence problem and the Bellman equation are equivalent.

Example: A consumer has an initial stock, w_0, of a good to consumer over her lifetime. Consider a value function

$$V_T(w_1) = \sum_{t=1}^{T} \beta^{t-1} u(c_t^*).$$

It follows that

$$V_{T+1}(w_0) = \sum_{t=0}^{T} \beta^t u(c_t)$$
$$V_{T+1}(w_0) = u(c_0) + \beta \sum_{t=1}^{T} \beta^{t-1} u(c_t)$$
$$V_{T+1}(w_0) = \max_{c_0, w_1} u(c_0) + \beta V_T(w_1),$$

subject to $w_1 = w_0 - c_0$. By substitution, you can find

$$V_{T+1}(w_0) = \max_{w_1} u(w_0 - w_1) + \beta V_T(w_1).$$

The first–order condition is

$$-u'(w_0 - w_1) + \beta V_T'(w_1) = 0.$$

Note that

$$V_T(w_1) = u(c_1) + \beta u(c_2) + \dots$$
$$V_T(w_1) = u(w_1 - w_2) + \beta u(w_2 - w_3) + \dots$$

Thus, Returning to the first–order condition, from the envelope theorem you can find that

$$-u'(w_0 - w_1) + \beta(c_1) = 0.$$

So,

$$u'(w_0 - w_1) = \beta u'(w_1 - w_2)$$
$$u'(c_0) = \beta u'(c_1).$$

2.4.1. The McCall Search Model

Consider a worker that is searching for employment. Each period the worker is offered a wage, $w \in [0, B]$, that may be accepted or rejected. If the worker accepts the offer, then the worker works at the wage, w, for perpetuity. If the worker rejects the offer, then the worker receives unemployment benefits, c, for the current period and receives another offer in the subsequent period. The value function for the unemployed worker is

$$V(w) = \max_{\{\text{accept, reject}\}} \left\{ \frac{w}{1-\beta}, c + \beta \int_0^B V(w')\,\mathrm{d}F(w') \right\}.$$

Define the reservation wage, $\overline{w}$, to be the wage where the worker is indifferent between accepting and rejecting the wage offer. At the reservation wage, $\overline{w}$, the value of accepting an offer and the value of rejecting an offer are equal

$$\frac{\overline{w}}{1-\beta} = c + \beta \int_0^B V(w')\,\mathrm{d}F(w').$$

It follows that

$$\frac{\overline{w}}{1-\beta}\left[\int_0^{\overline{w}} (1)\,\mathrm{d}F(w') + \int_{\overline{w}}^B (1)\,\mathrm{d}F(w')\right] = c + \beta \int_0^{\overline{w}} \frac{\overline{w}}{1-\beta}\,\mathrm{d}F(w') + \beta \int_{\overline{w}}^B \frac{w'}{1-\beta}\,\mathrm{d}F(w')$$

$$\overline{w} \int_0^{\overline{w}} (1)\,\mathrm{d}F(w') + \overline{w} \int_{\overline{w}}^B (1)\,\mathrm{d}F(w') = c + \frac{1}{1-\beta} \int_{\overline{w}}^B (\beta w' - \overline{w})\,\mathrm{d}F(w') + \overline{w} \int_{\overline{w}}^B \frac{w'}{1-\beta}\,\mathrm{d}F(w')$$

$$\overline{w} - c = \frac{\beta}{1-\beta} \int_{\overline{w}}^B w'\,\mathrm{d}F(w') - \frac{\overline{w}}{1-\beta} \int_{\overline{w}}^B (1)\,\mathrm{d}F(w') + \frac{(1-\beta)\overline{w}}{1-\beta} \int_{\overline{w}}^B (1)\,\mathrm{d}F(w')$$

$$\overline{w} - c = \frac{\beta}{1-\beta} \int_{\overline{w}}^B w'\,\mathrm{d}F(w') - \beta\overline{w} \int_{\overline{w}}^{\beta} (1)\,\mathrm{d}F(w')$$

You can deduce that the opportunity cost of searching is equal to the gain to searching for a wage above the reservation wage, $\overline{w}$,

$$\underbrace{\overline{w} - c}_{\text{Opportunity cost of searching}} = \underbrace{\frac{\beta}{1-\beta} \int_{\overline{w}}^B (w' - \overline{w})\,\mathrm{d}F(w')}_{\text{Gain to searching for a wage above } \overline{w}} \equiv h(w).$$

Note that the gain to searching for a wage above $\overline{w}$ as a function of w is decreasing at a decreasing rate

$$h'(w) < 0$$
$$h''(w) < 0,$$

and

$$h(0) = \frac{\beta}{1-\beta}\,\mathbb{E}[w]$$
$$h(B) = 0.$$

2.4.2. A Finite Horizon Dynamic Cake Eating Problem

Consider a fat kid who is endowed with a cake of size W at time $t = 1$, who has a finite time, $t = 1, \dots, T$, to eat the cake. The flow of utility is $u(c_t)$, where

$$\begin{aligned} u'(c_t) &> 0 \\ u''(c_t) &< 0 \\ \lim_{c_t \to 0} u'(c_t) &= \infty \\ \lim_{c_t \to \infty} u'(c_t) &= 0. \end{aligned}$$

The lifetime utility is

$$U = \sum_{t=1}^{T} \beta^{t-1} u(c_t),$$

where there is a discount factor $0 < \beta < 1$. The resource constraint is

$$W_{t+1} = W_t - c_t.$$

The sequence problem is

$$\begin{aligned} \max_{\{c_t, W_{t+1}\}_{t=1}^{T}} \; & U = \sum_{t=1}^{T} \beta^{t-1} u(c_t) \\ \text{s. t. } & W_{t+1} = W_t - c_t. \end{aligned}$$

Note that the resource constraint implies

$$\begin{aligned} W_{T+1} &= W_T - c_T \\ W_T &= W_{T+1} + c_T \\ W_T &= W_{T-1} - c_{T-1} \\ W_{T-1} &= W_{T+1} + c_T + c_{T-1} \\ &\vdots \\ W_1 &= W_{T+1} + \sum_{t=1}^{T} c_t \end{aligned}$$

Therefore, the Lagrangian can be written as

$$\mathscr{L} = \sum_{t=1}^{T} \beta^{t-1} u(c_t) - \lambda \left(W_{T+1} - W_1 + \sum_{t=1}^{T} c_t \right) + \phi W_{T+1},$$

where λ is the multiplier on the resource constraint and represents the shadow value to loosing the constraint, and ϕ is the multiplier on the terminal inequality constraint. Note that W_1 is given and if

$$W_{T+1} > 0 \Leftrightarrow \phi = 0.$$

The first–order conditions are

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial c_t} &= \beta^{t-1} u'(c_t) - \lambda = 0 \\ \frac{\partial \mathscr{L}}{\partial W_{T+1}} &= -\lambda + \phi = 0, \end{aligned}$$

for $t = 1, \ldots, T$. Note that if there is consumption, $c_t > 0$, then

$$\beta^{t-1} u'(c_t) = \lambda > 0.$$

Therefore, the multiplier on the terminal inequality constraint is positive

$$\phi = \lambda > 0,$$

and all the cake is consumed in the finite time, T,

$$W_{T+1} = 0.$$

You can find that

$$\beta^{t-1} u'(c_t) = \beta^t u'(c_{t+1}),$$

for $t = 1, \ldots, T-1$. Thus, the Euler equation is

$$u'(c_t) = \beta u'(c_{t+1}).$$

Given a function form for utility, you can solve $T-1$ Euler equations and the budget constraint, $W_1 = \sum_{t=1}^{T} c_t$, to find the optimal consumption plan.

2.4.3. An Infinite Horizon Dynamic Cake Eating Problem

Consider a fat kid who is endowed with a cake of size W in an initial period $t = 0$, who has infinite time to eat the cake. The optimization problem is

$$\begin{aligned} \max_{\{c_t, W_{t+1}\}_{t=1}^{T}} U &= \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s. t. } W_{t+1} &= W_t - c_t \\ W(0) &= W_0. \end{aligned}$$

The Bellman equation for the problem is

$$\begin{aligned} V(W) &= \max_{\{c,W\}} u(c) + \beta V(W') \\ \text{s. t. } W' &= W - c, \end{aligned}$$

where W is the state variable and c is the control variable. You can now solve the Bellman equation by finding the first–order condition

$$-u'(c) + \beta V'(w') = 0,$$

and using the envelope theorem after noting

$$\begin{aligned} V(W) &= u(W - W') + \beta V(W') \\ \frac{\partial V(W)}{\partial W} &= V'(W) = u'(W - W') = u'(c). \end{aligned}$$

You can then find the Euler equation to be

$$u'(c) = \beta u'(c').$$

The solution is

$$c = g(W),$$

and you solve for $g(W)$ from noting

$$\begin{aligned} u'(g(W)) &= \beta u'(g(W')) \\ u'(g(W)) &= \beta u'(g(W - g(W))). \end{aligned}$$

Suppose that you think that utility is logarithmic. You can use the method of undetermined coefficients to solve for $g(W)$. You can then make a good guess for the functional form of the value function

$$V(W) = A + B\ln(W).$$

You can then plug the guess into the Bellman equation

$$V(W) = \max_{W'} \ln(W - W') + \beta(A + B\ln(W')).$$

The Euler equation is

$$\frac{1}{W - W'} = \frac{\beta B}{W'}.$$

You can then solve for the optimal state transition

$$\begin{aligned} W' &= \beta B(W - W') \\ W' + \beta BW' &= \beta bW \\ W'(1 + \beta B) &= \beta BW \\ W' &= \frac{\beta BW}{1 + \beta B}. \end{aligned}$$

Optimal consumption is given by

$$\begin{aligned} c &= W - W' \\ c &= W - \frac{\beta BW}{1 + \beta B} \\ c &= \left(1 - \frac{\beta B}{1 + \beta B}\right)W \\ c &= \left(\frac{1 + \beta B}{1 + \beta B} - \frac{\beta B}{1 + \beta B}\right)W \\ c &= \frac{1}{1 + \beta B}W. \end{aligned}$$

You can now solve for the parameters by substituting the optimal consumption and state transition into the Bellman equation

$$\begin{aligned} A + B\ln(W) &= \ln\left(\frac{1}{1 + \beta B}W\right) + \beta\left(A + B\ln\left(\frac{\beta BW}{1 + \beta B}\right)\right. \\ A + B\ln(W) &= \ln\left(\frac{1}{1 + \beta B}\right) + \ln(W) + \beta A + \beta B\ln\left(\frac{\beta B}{1 + \beta B}\right) + \beta B\ln(w). \end{aligned}$$

Notice that it is implied

$$\begin{aligned} A &= \ln\left(\frac{1}{1 + \beta B}\right) + \beta A + \beta B\ln\left(\frac{\beta B}{1 + \beta B}\right) \\ B &= 1 + \beta B. \end{aligned}$$

You can now solve for the parameters and note that

$$\begin{aligned} B(1-\beta) &= 1 \\ B &= \frac{1}{1-\beta}. \end{aligned}$$

You can substitute the parameter into the policy equation to find

$$\begin{aligned} c &= \frac{1}{1+\beta B} W \\ c &= \frac{1}{1+\beta \frac{1}{1-\beta}} W \\ c &= \frac{1}{\frac{1-\beta}{1-\beta} + \frac{\beta}{1-\beta}} W \\ c &= \frac{1}{\frac{1}{1-\beta}} W. \end{aligned}$$

Thus the optimal policy function for consumption is

$$g(W) \equiv (1-\beta)W.$$

Quarter III

"It is useless, after all, to complain against inexorable reality."
– Jack Vance, *Cugel's Saga*

The goal of this section is to cover topics in Macroeconomics that every economics graduate student should know. The two main topics covered are are vector autoregressions (VARs) and New Keynesian Dynamic Stochastic General Equilibrium models (NK DSGE models).

3.1. Reduced-Form and Structural Vector Autoregressions

Reading: Stock, James, and Mark Watson (2001). "Vector Autoregressions," *Journal of Economic Perspectives* 15, 101-115.

The advantages to VAR models are that they are atheoretical, flexible, and can fit any frequency data. The major pitfall to VAR models is overfitting the model with regressors.

3.1.1. Reduced–Form Vector Autoregressions

Assume a system of linear equations for output, y, inflation, π, and the interest rate, i,

$$\begin{aligned}
y_t &= \alpha_y + \cdots + \beta_j y_{t-j} + \cdots + \gamma_j \pi_{t-j} + \cdots + \delta_j i_{t-j} + \mu_t^y \\
\pi_t &= \alpha_\pi + \cdots + \theta_j y_{t-j} + \cdots + \phi_j \pi_{t-j} + \cdots + \lambda_j i_{t-j} + \mu_t^\pi \\
i_t &= \alpha_i + \cdots + \psi_j y_{t-j} + \cdots + \kappa_j \pi_{t-j} + \cdots + \rho_j i_{t-j} + \mu_t^i,
\end{aligned}$$

that can be written in matrix form as

$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = \begin{pmatrix} \alpha_y \\ \alpha_\pi \\ \alpha_i \end{pmatrix} + A_1 \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + \cdots + A_j \begin{pmatrix} y_{t-j} \\ \pi_{t-j} \\ i_{t-j} \end{pmatrix} + \begin{pmatrix} \mu_t^y \\ \mu_t^\pi \\ \mu_t^i \end{pmatrix}.$$

Let

$$A(L) = A_0 + A_1 L + A_2 L^2 + \cdots + A_k L^k,$$

where

$$L^j x_t \equiv x_{t-j}.$$

Then a reduced–form vector autoregression (VAR) is written

$$\vec{x}_t = \vec{\alpha} + A(L)\vec{x}_{t-1} + \vec{\mu}_t,$$

where A is a $n \times k$ matrix of scalar lag polynomials of any number of lags k.

You can estimate

$$y_t = x_t \vec{\beta} + \mu_t,$$

where

$$\begin{aligned}
x_t &= [1 \ldots y_{t-k} \ldots \pi_{t-k} \ldots i_{t-k}] \\
\vec{\beta}\,' &= [\alpha_y \beta_1 \ldots \beta_k \ \ldots \alpha_z \lambda_1 \ldots \lambda_k].
\end{aligned}$$

by OLS

$$\hat{\beta} = (x'x)^{-1} x'y.$$

The estimates are consistent, but biased because μ_t is not independent of x_t.

Proof. Note that

$$\begin{aligned}
\hat{\beta} &= (x'x)^{-1}x'(x\beta + \mu) \\
\hat{\beta} &= \beta + (x'x)^{-1}x'\mu \\
\mathbb{E}[\hat{\beta}] &= \beta + \mathbb{E}[(x'x)^{-1}x'\mu] \\
\operatorname*{plim}_{T\to\infty} \hat{\beta} &= \beta + \operatorname*{plim}_{T\to\infty} (x'x)^{-1}x'\mu \\
\operatorname*{plim}_{T\to\infty} \hat{\beta} &= \beta + \operatorname*{plim}_{T\to\infty} (\tfrac{x'x}{T})^{-1} \operatorname*{plim}_{T\to\infty} (\tfrac{x'\mu}{T}),
\end{aligned}$$

and

$$\tfrac{1}{T}x'x = \tfrac{1}{T}\sum_{t=0}^{T} x_t' x_t \overset{\mathrm{Pr}}{\to} Q.$$

Thus, the estimates are biased. However,

$$\tfrac{1}{T}x'\mu = \tfrac{1}{T}\sum_{t=0}^{T} x_t' \mu_t \overset{\mathrm{Pr}}{\to} 0.$$

So,

$$\begin{aligned}
\operatorname*{plim}_{T\to\infty} \hat{\beta} &= \beta + Q^{-1}(0) \\
\operatorname*{plim}_{T\to\infty} \hat{\beta} &= \beta.
\end{aligned}$$

Therefore, the estimates are consistent. □

Sims, Stock, and Watson (1990) show that coefficients in VAR models with nonstationary variables are consistent as long as the test statistic is not solely based on nonstationary variables. Also, note that the bias is larger for higher order VAR models. The variance of μ_t is $\mathbb{V}[\mu_t] = \Omega$. However, typically, $\operatorname{corr}(\mu_t^i, \mu_t^y) \neq 0$.

Example: Forecasting with a Reduced–Form Vector Autoregression
Consider the reduced–form model VAR(1) model of output, y, inflation, π, and the interest rate, i,

$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = \vec{\alpha} + A(L) \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + \vec{\mu}_t.$$

The expected value of the next period is

$$\begin{aligned}
\mathbb{E}_t \begin{pmatrix} y_{t+1} \\ \pi_{t+1} \\ i_{t+1} \end{pmatrix} &= \mathbb{E}_t \left[\vec{\alpha} + A(L) \begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} + \vec{\mu}_{t+1} \right] \\
\mathbb{E}_t \begin{pmatrix} y_{t+1} \\ \pi_{t+1} \\ i_{t+1} \end{pmatrix} &= \vec{\alpha} + A(L) \begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} + \mathbb{E}_t \, \vec{\mu}_{t+1}.
\end{aligned}$$

The exogenous shock cannot be explained, $\mathbb{E}_t[\vec{\mu}_{t+1}] = 0$. Then

$$\mathbb{E}_t \begin{pmatrix} y_{t+1} \\ \pi_{t+1} \\ i_{t+1} \end{pmatrix} = \vec{\alpha} + A(L) \begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix}.$$

The best forecast leaves only the exogenous shock unexplained. You can then use the forecast the next period and repeat iteratively.

3.1.2. Structural Vector Autoregressions

Now, consider a structural vector autoregression representation

$$\vec{x}_t = \vec{\alpha} + B(L)\vec{x}_{t-1} + S\vec{\varepsilon}_t.$$

For the system of linear equations for output, y, inflation, π, and the interest rate, i, the structural VAR is

$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = \vec{\alpha} + B(L) \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + \begin{pmatrix} S \end{pmatrix} \begin{pmatrix} \varepsilon_t^y \\ \varepsilon_t^\pi \\ \varepsilon_t^i \end{pmatrix}.$$

If

$$\mathbb{V}\left(\vec{\varepsilon}_t \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$S \cdot \vec{\varepsilon}_t = \vec{\mu}_t.$$

Note that

$$\begin{aligned} \mathbb{V}[\vec{\mu}]_t &= S \cdot \mathbb{V}[\vec{\varepsilon}_t] \cdot S' \\ \Omega_\mu &= S\Omega_\varepsilon S' \\ \Omega_\mu &= SS', \end{aligned}$$

and the reduced–form VAR does not identify S. If S has 9 elements, then Ω_μ needs 6 restrictions.

Let $S_\circ$ satisfy $S_\circ S_\circ' = \Omega_\mu$ and let U be any 3×3 orthogonal matrix (rotation matrix). Then

$$\begin{aligned} (S_\circ U') \cdot (S_\circ U')' &= S_\circ U \cdot U \cdot S_\circ' \\ (S_\circ U') \cdot (S_\circ U')' &= S_\circ S_\circ' \\ (S_\circ U') \cdot (S_\circ U')' &= \Omega_\mu \end{aligned}$$

3.2. Methods of Identifying Structural VARs

Reading: Christiano, Lawrence, Martin Eichenbaum, and Charles Evans (1999). "Monetary Policy Shocks: What Have We Learned and to What End?" *Handbook of Macroeconomics* 1, 65-148.

3.2.1. Recursive Identification

Suppose that inflation, π_t, is a slow variable and cannot respond within a month to changes in output, y_t, or interest rate, i_t. Suppose that output, y_t, can only respond to inflation, π_t. Suppose that the interest rate, i_t, is a fast variable and can respond to changes in inflation, π_t, and output, y_t. You can implement a recursive identification of S by writing the system as

$$\begin{aligned}
\vec{x}_t &= \vec{\alpha} + B(L)\vec{x}_{t-1} + S\vec{\varepsilon}_t \\
S^{-1}\vec{x}_t &= S^{-1}\vec{\alpha} + S^{-1}B(L)\vec{x}_{t-1} + \vec{\varepsilon}_t \\
S^{-1}\vec{x}_t - S^{-1}B(L)\vec{x}_{t-1} &= S^{-1}\vec{\alpha} + \vec{\varepsilon}_t \\
S^{-1}\vec{x}_t - S^{-1}B_0\vec{x}_{t-1} - \cdots - S^{-1}B_{k-1}\vec{x}_{t-k} &= S^{-1}\vec{\alpha} + \vec{\varepsilon}_t \\
A(L)\vec{x}_t &= \vec{\gamma} + \vec{\varepsilon}_t,
\end{aligned}$$

where

$$A(L) = A_0 + A_1L + \cdots + A_kL^k.$$

It follows that

$$\begin{aligned}
A_0 &= S^{-1} \\
A_1 &= -S^{-1}B_0 = -A_0B_0 \\
A_k &= -S^{-1}B_{k-1} = -A_0B_{k-1},
\end{aligned}$$

and you can write the system as

$$A_0 \begin{pmatrix} \pi_t \\ y_t \\ i_t \end{pmatrix} = \vec{\gamma} + (A_1L + \cdots + A_kL^k) \begin{pmatrix} \pi_t \\ y_t \\ i_t \end{pmatrix} + \vec{\varepsilon}_t.$$

Note that if A_0 is lower triangular, then S is lower triangular from Cramer's rule for inverse matrices. The assumptions that inflation is a slow variable, output responds only to the interest rate, and the interest rate is a fast variable allow the system to be written as

$$\begin{pmatrix} a & 0 & 0 \\ e & f & 0 \\ b & c & d \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \\ i_t \end{pmatrix} = \vec{\gamma} + (A_1 + A_2L + A_3L^2 + \cdots + A_kL^{k-1}) \begin{pmatrix} \pi_{t-1} \\ y_{t-1} \\ i_{t-1} \end{pmatrix} + \vec{\varepsilon}_t.$$

Furthermore,

$$S = A_0^{-1}.$$

You can now perform a Cholesky decomposition of Ω_μ to obtain an unique lower triangular matrix such that

$$SS' = \Omega_\mu.$$

Through Gaussian elimination

$$L\Omega_\mu = U,$$

if the determinant of the principle diagonals is not zero. The inverse of L is also triangular, so

$$\Omega_\mu = L^{-1}U.$$

If Ω_μ is symmetric, then

$$\Omega'_\mu = (L^{-1}U)' = U'L^{-1\prime}.$$

Because $\Omega'_\mu = \Omega_\mu$

$$\begin{aligned} L^{-1\prime} &= U \\ U' &= L^{-1}. \end{aligned}$$

Thus,

$$\Omega_\mu = SS',$$

where S is an unique lower triangular matrix.

Example: Block Recursive Identification

Suppose that you cannot fully identify S, but you can identify the effects of monetary policy shocks. That is, S is not lower triangular. Then A_0 is block lower triangular if and only if S is block lower triangular. For example, consider

$$S = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ s_{31} & s_{32} & s_{33} \end{pmatrix}.$$

The impact effect of a monetary policy shock is

$$\begin{aligned} \vec{\mu}_t &= S\vec{\varepsilon}_t \\ \vec{\mu}_t &= S\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \vec{\mu}_t &= \vec{S}_3. \end{aligned}$$

Up to scale, $\vec{S}_3$ is identified. Typically, analysis is performed for a $x\%$ shock. For example, a 0.25% shock to the interest rate is

$$\vec{\varepsilon}_t = \begin{pmatrix} 0 \\ 0 \\ \frac{0.25}{s_{33}} \end{pmatrix},$$

and s_{33} drops out.

Note that if the shock to the fast variable is all that you are interested in, then in practice you can estimate impulse response functions using any fully recursive ordering by utilizing a Cholesky decomposition.

3.2.2. Long-Run Restrictions

Reading: Blanchard, Olivier, and Danny Quah (1989). "The Dynamic Effects of Aggregate Demand and Supply Disturbances," *American Economic Review* 79, 655-673.

Reading: Galí , Jordi (1999). "Technology, Employment, and the Business Cycle: Do Technology Shocks Explain Aggregate Fluctuations?" *American Economic Review* 89, 249-271.

If you are interested in aggregate demand and aggregate supply shocks, then you may consider a VAR model with the growth in GDP, $\Delta \log y$, and unemployment, u. You can write the model as

$$\begin{pmatrix} \Delta \log y_t \\ u_t \end{pmatrix} = \vec{\alpha} + B(L) \begin{pmatrix} \Delta \log y_{t-1} \\ u_{t-1} \end{pmatrix} + \vec{\nu}_t,$$

where

$$\vec{\nu}_t = S\vec{\varepsilon}_t.$$

Note that S is a 2×2 matrix that needs 3 identifying restrictions to identify the non–duplicate elements of Ω_ν such that

$$SS' = \Omega_n u.$$

You must use a stationary VAR model. You can then invert the VAR so that

$$\begin{pmatrix} \Delta \log y_t \\ u_t \end{pmatrix} = \vec{\gamma} + \vec{\nu}_t + C_1 \vec{\nu}_{t-1} + \ldots,$$

where $\vec{V}_t = S\vec{\varepsilon}_t$. You can limit the long–run effect on $\vec{\varepsilon}_t$ so that

$$\begin{pmatrix} \Delta \log y_t \\ u_t \end{pmatrix} = \lim_{j \to \infty} C_j S = 0.$$

As $t \to \infty$, then

$$\log y_t = \sum_{j=1}^{\infty} (\Delta \log y_{t+j})$$

$$\log y_t = \begin{pmatrix} 1 & 0 \end{pmatrix} \sum_{j=1}^{\infty} C_j S \vec{\varepsilon}_t.$$

The identifying assumption is that the long–run effect of the aggregate demand shock, $\varepsilon_{d,t}$, on the growth in GDP, $\Delta \log y_t$, is 0. Thus, the fourth restriction on S is that

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \sum_{j=1}^{\infty} C_j S \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.$$

Example: Suppose that you are interested in the growth rate of labor productivity, x_t, and labor hours, n_t. You can write a reduced–form VAR as

$$\begin{pmatrix} \Delta \log x_t \\ \Delta \log n_t \end{pmatrix} = \vec{\alpha} + B(L) \begin{pmatrix} \Delta \log x_{t-1} \\ \Delta \log n_{t-1} \end{pmatrix} + \vec{\mu}_t,$$

where

$$\vec{\mu}_t = S \begin{pmatrix} \varepsilon_{x,t} \\ \varepsilon_{n,t} \end{pmatrix}.$$

The identifying assumption is that $\varepsilon_{n,t}$ has no long–run effect on the level of hours worked.

You can invert the VAR so that it can be written as

$$\begin{pmatrix} \Delta \log x_t \\ \Delta \log n_t \end{pmatrix} = \vec{\gamma} + S\vec{\varepsilon}_t + C_1 S \vec{\varepsilon}_{t-1} + \ldots.$$

It follows that

$$\begin{aligned} \vec{x}_t &= A\vec{x}_{t-1} + \vec{\mu}_t \\ (I - AL)\vec{x}_t &= \vec{\mu}_t \\ \vec{x}_t &= (I + AL + A^2L^2 + \ldots)\vec{\mu}_t. \end{aligned}$$

Note that

$$\log n_t = \sum_{t=1}^{T} \Delta \log n_t + \log n_0.$$

Assuming that there is no long–run effect of the shock, $\varepsilon_{n,t}$, on $\log n_t$, then

$$\sum_{t=1}^{T} \Delta \log n_t = 0 \text{ as } t \to \infty.$$

The long–run restriction is

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \left(\sum_{j=0}^{\infty} C_j \right) S \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0,$$

where $C_0 = I$. Thus, 1 additional linear restriction on S is provided by the identifying assumption.

Faust and Leeper (1997) point out a caveat that the long–run effects of a VAR model are not precisely estimated. This is because of the uncertainty of $B(L)$ implies uncertainty of long–run effects. Furthermore, identification of S is not robust even in large samples.

3.2.3. Sign Restrictions

Reading: Uhlig, Harald (2005). "What Are the Effects of Monetary Policy on Output? Results from an Agnostic Identification Procedure." *Journal of Monetary Economics* 52, 381–419.

Reading: Baumeister, Christiane, and James Hamilton (2015). "Sign Restrictons, Strucutral Vector Autoregressions, and Useful Prior Information," *Econometrica* 83, 1963-1999.

Sign restrictions are priors for the direction of the effects produced by exogenous shocks. Consider a uniform HAAR prior on μ. The prior is not flat over the elasticities in the model. After the sign restrictions are imposed, the prior for μ is a truncated Cauchy distribution. The Bayesian approach is to impose a prior on the coefficients, $B(L)$, such as a Minnesota prior.

Example: Consider the system

$$\vec{x}_t = \vec{\alpha} + B(L)\vec{x}_{t-1} + S\vec{\varepsilon}_t,$$

that does not identify S in general. You can impose sign restrictions on the impulse response function to identify some set $\mathbb{S}$ of admissible S matrices. For example

$$\mathbb{S} = \{S : SS' = \Omega_\mu ; s_{13} \leq 0\},$$

where a two period sign restriction on the impulse response function implies

$$B_0 S_{13} \leq 0.$$

Example: Set Identification
Suppose that you want to compute the IRF for every $S \in \mathbb{S}$. To compute $\mathbb{S}$, first draw matrices S randomly such that $SS' = \Omega_\mu$. You can let C be the Cholesky factorization of Ω_μ such that $C'C = \Omega_\mu$. Let $S = CU$ for some orthogonal matrix U, so,

$$SS' = CUU'C' = CC'.$$

Note that if $SS' = \Omega_\mu$, then there exists a matrix U such that $S = CU$.

Proof.

$$SS' = \Omega_\mu = CC'$$
$$S = CC'(S')^{-1}$$

You can show that $C'(S')^{-1}$ is orthogonal. That is

$$\begin{aligned}
(C'(S')^{-1})'C'(S')^{-1} &= I \\
((S')^{-1})'CC'(S')^{-1} &= I \\
S^{-1}CC'(S')^{-1} &= I \\
S^{-1}\Omega_\mu(S')^{-1} &= I \\
S^{-1}\Omega_\mu(S')^{-1}S^{-1}S &= I \\
S^{-1}\Omega_\mu(S'S)^{-1}S &= I \\
S^{-1}\Omega_\mu(\Omega_\mu)^{-1}S &= I \\
S^{-1}S &= I.
\end{aligned}$$

$\square$

You can draw matrices U in a uniform way using a HAAR measure. First draw every element of M randomly from a standard normal distribution. Then compute $M = QR$, where Q is an orthogonal matrix.

3.2.4. External Instruments

Reading: Stock, James, and Mark Watson (2012). "Disentangling the Channels of the 2007-09 Recession," *Brookings Papers on Economic Activity*, Spring, 81-135.

Reconsider the system

$$\vec{x}_t = \vec{\alpha} + B(L)\vec{x}_{t-1} + S\vec{\varepsilon}_t,$$

that can be inverted and written as

$$A(L)\vec{x}_t = \vec{\gamma} + \vec{\varepsilon}_t,$$

where A_0 are simultaneous effects. It used to be common to use internal instruments to estimate elements of S.

Example: Suppose that inflation, π_t, is slow, output, y_t, is of interest, and the interest rate, i_t, is a fast variable. Then, $\mu_{\pi,t}$ is independent of $\varepsilon_{i,t}$ and $\varepsilon_{y,t}$. Note that

$$\begin{aligned}\vec{\mu}_t &= S\vec{\varepsilon}_t \\ \mu_{i,t} &= s_{31}\varepsilon_{\pi,t} + s_{32}\varepsilon_{y,t} + s_{33}\varepsilon_{i,t}.\end{aligned}$$

You can use an instrumental variable (IV) regression with $\mu_{\pi,t}$ as an instrument to estimate s_{31}.

The next method is a narrative approach where you can identify exogenous shocks. Let z_t be any instrument for $\varepsilon_{j,t}$. This requires that the instrument is exogenous and relevant.

- Exogeneity:
$$\mathbb{E}[z_t\varepsilon_{k,t}] = 0, \text{ where } k \neq j.$$
- Relevance:
$$E[z_t\varepsilon_{j,t}] \neq 0.$$

You can then regress $\mu_{n,t}$ on z_t. This implies that the coefficient is

$$\begin{aligned}\beta &= \frac{\mathbb{E}[\mu_{n,t}z_t]}{\mathrm{Var}[z_t]} \\ \beta &= \frac{\mathbb{E}[s_n\vec{\varepsilon}_t z_t]}{\mathrm{Var}[z_t]} \\ \beta &= \frac{s_n\,\mathbb{E}[\vec{\varepsilon}_t z_t]}{\mathrm{Var}[z_t]} \\ \beta &= \frac{s_n(0\ldots\theta\ldots0)'}{\mathrm{Var}[z_t]} \\ \beta &= \frac{\theta s_{nj}}{\mathrm{Var}[z_t]}.\end{aligned}$$

Thus, the j^{th} column of S is identified up to scale $\frac{\theta}{\mathrm{Var}[z_t]}$. Therefore, you can normalize the scale so that you can consider the effects from an $x\%$ shock.

3.2.5. High-Frequency Identification

Reading: Cochrane, John, and Monika Piazzesi (2002). "The Fed and Interest Rates: A High-Frequency Identification," *American Economic Review* 92, 90-95.

Reading: Fisher, Jonas, and Ryan Peters (2010). "Using Stock Returns to Identify Government Spending Shocks," *Economic Journal* 120, 414-436.

Reading: Gertler, Mark, and Peter Karadi (2015). "Monetary Policy Surprises, Credit Costs, and Economic Activity," *American Economic Journal: Macroeconomics* 7, 44-76.

Suppose that you look closely at the change in the interest rate, i_t, around Federal Open Market Committee (FOMC) announcements. The high frequency response is correlated with monthly $\varepsilon_{i,t}$, but not $\varepsilon_{i,t}$ generally. That is, $\varepsilon_{i,t}$ is the sum of all the effects and shocks to the interest rate, i_t, over the month t. The instrument, z_t, is the high frequency response of the 1–year treasury bond yield to the FOMC announcement. The instrument is relevant, $E[z_t \varepsilon_{i,t}] \neq 0$, and exogenous, $\mathbb{E}[z_t \varepsilon_{k,t}] = 0$, where $k \neq i$. You can regress $\vec{\mu}_t$ on z_t to estimate $\vec{s}_3$, if the interest rate, i_t is a fast variable.

3.2.6. Impulse Response Functions

Once you have identified S, then

$$\vec{\mu}_t = S\vec{\varepsilon}_t = \begin{pmatrix} \vec{s}_1 & \vec{s}_2 & \vec{s}_3 \end{pmatrix} \begin{pmatrix} \varepsilon_{\pi,t} \\ \varepsilon_{y,t} \\ \varepsilon_{i,t} \end{pmatrix}.$$

The impact effect of $\varepsilon_{i,t}$ on $\vec{x}_t$ is $\vec{s}_3$. Remember that

$$\vec{x}_t = \vec{\alpha} + B(L)\vec{x}_{t-1} + S\vec{\varepsilon}_t.$$

The recursive assumption says that

$$\vec{s}_3 = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}.$$

The impact effect of $\varepsilon_{i,t}$ on the interest rate i_t is a, the impact effect of $\varepsilon_{i,t}$ on output y_t is 0, and the impact effect of $\varepsilon_{i,t}$ on inflation π_t is 0. If you want a $x\%$ interest rate shock, then let $\varepsilon_{i,t} = \frac{x}{s_{33}}$.

Definition: Impulse Response Function
An impulse response function is the effect of a shock to $\vec{x}_t$ after j periods

$$\text{IRF}(\varepsilon_t, j) = \mathbb{E}[\vec{x}_{t+j}|\vec{x}_{t-1}, \varepsilon_t = (0\ 0\ 1)'] - \mathbb{E}[\vec{x}_{t+j}|\vec{x}_{t-1}, \vec{\varepsilon}_t = (0\ 0\ 0)'].$$

In this case,

$$\vec{\mu}_t = S(0\ 0\ 1)' = \vec{s}_3.$$

In general, the initial conditions in period t do not matter. The impulse response function is $S\vec{\varepsilon}_t$ in the period of the shock, $B_0 S\vec{\varepsilon}_t$ in the second period, $B_1 S\vec{\varepsilon}_t + B_0(B_0 S\vec{\varepsilon}_t)$ in the third period, and you can continue iteratively.

The caveats to estimating impulse response functions (IRFs) are that the recursive ordering is not always obvious, there are often more than one fast variables, data may be released with a short frequency, slow variables are assumed not to respond within a given period, and the structural shocks to slow variables are problematic.

3.2.7. Vector Autoregression Extensions

First–order Companion Form

You can rewrite any VAR(k) model as a VAR(1). For example, let

$$\vec{x}_t = \vec{\alpha} + A(L)\vec{x}_{t-1} + \vec{\mu}_t,$$

where

$$\underset{\substack{\uparrow \\ nk\times 1}}{\begin{pmatrix} x_t \\ \vdots \\ \vdots \\ x_{t-k+1} \end{pmatrix}} = \underset{\substack{\uparrow \\ nk\times 1}}{\begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}} + \underset{\substack{\uparrow \\ nk\times nk}}{\begin{pmatrix} A_0 & \cdots & \cdots & A_k \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{pmatrix}} \underset{\substack{\uparrow \\ nk\times 1}}{\begin{pmatrix} x_{t-1} \\ \vdots \\ \vdots \\ x_{t-k} \end{pmatrix}}.$$

The Invertability of an ARMA Process

Consider the process

$$a(L)y_t = b(L)\varepsilon_t.$$

You can invert the process so that

$$y_t = a(L)^{-1}b(L)\varepsilon_t.$$

Consider the process

$$\vec{z}_t = \vec{\alpha} + A\vec{z}_{t-1} + \mu_t.$$

It follows that

$$\begin{aligned} (I - AL)\vec{z}_t &= \vec{\alpha} + \mu_t \\ \vec{z}_t &= (I - AL)^{-1}(\vec{\alpha} + \mu_t). \end{aligned}$$

Note that

$$(I - AL)^{-1} = I + AL + \cdots + A^{\infty}L^{\infty}.$$

This process converges if and only if all eigenvalues of A are less than 1 in absolute value. Thus, any VAR can be written as a moving average process

$$\begin{aligned} \vec{z}_t &= (I + AL + \cdots + A^{\infty}L^{\infty})(\vec{\alpha} + \mu_t) \\ \vec{z}_t &= (I + AL + \cdots + A^{\infty}L^{\infty})\vec{\alpha} + (I + AL + \cdots + A^{\infty}L^{\infty})\vec{\mu}_t \\ \vec{z}_t &= (I - A)^{-1}\vec{\alpha} + (I + AL + \cdots + A^{\infty}L^{\infty})\vec{\mu}_t. \end{aligned}$$

The Lucas Critique: Reduced-form vs. Structural Models

Reading: Lucas, Robert (1976). "Econometric Policy Evaluation: A Critique," *Carnegie-Rochester Conference Series on Public Policy* 1, 19-46.

Robert Lucas criticized reduced–form VAR models. The foundation for his argument begins with the Phillip's curve structural model

$$y_t = y_t^* + \theta(P_t - P_t^e),$$

where P_t^e is $\mathbb{E}_t P_t$. Suppose that

$$P_t - P_{t-1} = \pi + \varepsilon_t,$$

where $\varepsilon \sim (0, \sigma^2)$. Then

$$P_t^e = P_{t-1} + \overline{\pi}.$$

It follows that the structural model is

$$\begin{aligned}
y_t &= y_t^* + \theta(P_t - (P_{t-1} + \overline{\pi})) \\
y_t &= y_t^* - \theta\overline{\pi} + \theta(P_t - P_{t-1}) \\
y_t &= (y_t^* - \theta\overline{\pi}) + \theta\pi_t.
\end{aligned}$$

The reduced–form model is of the form

$$y_t = \alpha + \beta\pi_t.$$

An econometrician estimates $\hat{\beta} = \theta > 0$, and argues that there is a positive effect of inflation on output. However, according to the structural model, the effect is ambiguous if if expectations for inflation, $\overline{\pi}$, also change. Modern macroeconomics begins with the assumption that agents are forward looking and form rational expectations of the future. Any model can be subject to the Lucas critique if its parameters are not invariant to the policy experiment being considered. This ultimately depends on the model and policy being considered. The Lucas critique does not argue that the model must be necessarily 'true', however, the parameters should be invariant to the policy under consideration if it is to provide reliable analysis.

3.3. Heterogeneity

Reading: Krusell, Per, and Anthony Smith (1998). "Income and Wealth Heterogeneity and the Macroeconomy," *Journal of Political Economy* 106, 867-896.

Reading: Gourinchas, Pierre-Oliver, and Jonathan Parker (2002). "Consumption over the Life Cycle," *Econometrica* 70, 47-89.

3.3.1. The Existence of a Representative Firm

Suppose that there are $n = 1, \ldots, N$ heterogeneous firms.

Definition: Aggregate Production Function
Define an aggregate production function as

$$Y = F(K, L),$$

where total capital, K, total labor, L, and total output, Y, are given by

$$K = \sum_{n=1}^{N} k_n \qquad L = \sum_{n=1}^{N} l_n \qquad Y = \sum_{n=1}^{N} y_n,$$

for firms $n = 1, \ldots, N$ with capital $k_1, \ldots, k_n$, labor $l_1, \ldots, l_n$, and output $y_1, \ldots, y_n$ respectively. There does not exist an aggregate production function

$$F(K, L) = \sum_{n=1}^{N} y_n,$$

unless all $f_n(k, l)$ are linear with the same slope.

Proof. Suppose that

$$\frac{\partial f_n}{\partial k} \neq \frac{\partial f_j}{\partial k},$$

for some $j \neq n$. Also, let

$$\frac{\partial f_n}{\partial k} > \frac{\partial f_j}{\partial k}.$$

Then it is possible to reallocate dk from j to n without changing aggregate capital, K, but increasing aggregate output, Y. Continue until

$$\frac{\partial f_n}{\partial k} = \frac{\partial f_j}{\partial k},$$

for all $j, n \in N$ and $k \in K$. Then it is also possible to reallocate dl from j to n until

$$\frac{\partial f_n}{\partial l} = \frac{\partial f_j}{\partial l},$$

for all $j, n \in N$ and $l \in L$. Now, let

$$\beta = \frac{\partial f_n}{\partial k},$$
$$\gamma = \frac{\partial f_n}{\partial l}.$$

Then it must be that

$$f_n(k, l) = \alpha_n + \beta k_n + \gamma l_n.$$

□

Let each firm n have production function $f_n(k,l)$ and profit functions $\pi_n(p,r,w)$, where p is the price, r is the rental price of capital, and w is the wage rate of labor. If the profit functions of the firms are homogeneous of degree 1, convex, monotonic in price, p, and strictly decreasing in rental price, r, and wage rate, w, then there is an aggregate profit function.

Definition: Aggregate Profit Function
Define an aggregate profit function as

$$\pi(p,r,w) = \sum_{i=1}^{N} \pi_n(p,r,w).$$

Proof. From Hotelling's Lemma, note that

$$\begin{aligned}\frac{\partial \pi_n}{\partial p} &= \quad y_n, \\ \frac{\partial \pi_n}{\partial r} &= -k_n, \\ \frac{\partial \pi_n}{\partial w} &= -l_n.\end{aligned}$$

You can check that

$$\begin{aligned}\frac{\partial \pi}{\partial p} &= \quad \sum_{i=1}^{N} \frac{\partial \pi_n}{\partial p} = \sum_{i=1}^{N} y_n = Y, \\ -\frac{\partial \pi}{\partial r} &= -\sum_{i=1}^{N} \frac{\partial \pi_n}{\partial r} = \sum_{i=1}^{N} k_n = K, \\ -\frac{\partial \pi}{\partial w} &= -\sum_{i=1}^{N} \frac{\partial \pi_n}{\partial w} = \sum_{i=1}^{N} l_n = L.\end{aligned}$$

Therefore, $\pi(p,r,w)$ is an aggregate production function. □

Theorem: If and only if you have perfectly competitive markets, then you can have perfect aggregation.

The conclusion is that the economy acts as if it is a single firm with profit function $\pi(p,r,w)$. You can then convert $\pi(p,r,w)$ to the production function $Y = F(k,L)$ through duality.

Given competitive markets and profit maximization, you can show that the economy behaves as if there is an aggregate production function with an aggregate production possibilities set

$$Y = \left\{ \sum_{i=1}^{N} y_n : y_n \in Y_n \right\},$$

but this does not prove the existence of an aggregate production function.

3.3.2. The Existence of a Representative Agent

Assume that there are commodities, $n = 1, \ldots, N$, and households, $h = 1, \ldots, H$, with utility, $U_n(\vec{c})$. The households each demand $\vec{c}_h(\vec{p}, m)$, where the houshold's income is m, commodities demanded are $\vec{c} \in \mathbb{R}^n$ and the respective prices are $\vec{p} \in \mathbb{R}^N$.

Definition: Aggregate Consumption Function
Define an aggregate consumption function as

$$\vec{C}(\vec{p}, \{m_h\}_{h=1}^H) = \sum_{h=1}^{H} \vec{c}_h(\vec{p}, m_h).$$

If you wish to restrict the distribution of $\{m_h\}_{h=1}^H$ so that you can represent aggregate consumption as $\vec{C}(\vec{p}, M)$, where $m = \sum_{h=1}^H m_h$, then it must be that reallocations of household income, m, do not affect consumption. It must be that

$$\frac{\partial \vec{c}_h(\vec{p}, m)}{\partial m} = \frac{\partial \vec{c}_j}{\partial m},$$

for all households, $h, j \in H$, and income levels, m. Thus, it must be that

$$\frac{\partial \vec{c}_h(\vec{p}, m)}{\partial m} = \vec{f}(\vec{p}),$$

where the marginal propensity to consume, $\vec{f}(\vec{p}$, is independent from household, h, and income level, m. So, it must be that consumption of every good is linear in m and has a linear Engel curve

$$\vec{c}_h(\vec{p}, m) = \vec{g}(\vec{p}) + \vec{f}(\vec{p})m.$$

Roy's Identity implies that

$$\vec{c}_h(\vec{p}, m) = -\frac{\partial V_h(\vec{p}, m)/\partial \vec{p}}{\partial V_h(\vec{p}, m)/\partial m},$$

where $V_h(\vec{p}, m)$ is the indirect utility function of household h. This implies that for an aggregate consumption function to exist, you need

$$-\frac{\partial V_h(\vec{p}, m)/\partial \vec{p}}{\partial V_h(\vec{p}, m)/\partial m} = \vec{g}(\vec{p}) + \vec{f}(\vec{p})m.$$

3.3.3. Gorman Aggregation

Definition: Representative Household
Heterogeneous agents act as if the economy is representative.

Theorem: The Gorman Theorem
Let

$$V_h(\vec{p}, m) = a_h(\vec{p}) + b(\vec{p})m.$$

Then $\vec{c}_h(\vec{p}, m)$ is linear in income, m, and the economy behaves as if there is a representative agent.

3.4. Balanced Growth and Real Business Cycles

Reading: King, Robert, Charles Plosser, and Sergio Rebelo (2002). "Production, Growth, and Business Cycles: Technical Appendix," *Computational Economics* 20, 87-116.

3.4.1. Balanced Growth

You may notice that the trend in a countries GDP is pretty constant. An economy is on a balanced growth path if output, Y, capital, K, labor, L, investment, I, and consumption, C, grow at constant rates

$$\gamma_Y = \frac{Y_{t+1}}{Y_t},\ \gamma_I = \frac{I_{t+1}}{I_t},\ \gamma_K = \frac{K_{t+1}}{K_t},\ \gamma_C = \frac{C_{t+1}}{C_t},\ \gamma_N = \frac{N_{t+1}}{N_t}.$$

Begin with the program of a representative household

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ &\text{s. t. } k_{t+1} = (1+r_t)k_t + w_t n_t - c_t, \end{aligned}$$

where c_t is the consumption of the household, l_t is leisure, and n_t is labor. The representative firm faces a constant returns to scale production function at each t

$$\begin{aligned} Y_t &= F(K_t, N_t, t) \\ \text{s. t. } K_{t+1} &= (1-\delta)K_t + I_t. \end{aligned}$$

The profit maximization program yields the real wage and real rate of capital

$$\begin{aligned} w_t &= \mathrm{MPL}_t \\ r_t &= \mathrm{MPK}_t - \delta. \end{aligned}$$

The resource constraints are

$$\begin{aligned} l_t + n_t &= 1 \\ C_t + I_t &\leq Y_t \\ C_t, L_t, N_t, K_t &\geq 0. \end{aligned}$$

The existience of a balanced growth path implies the following.

- $F(K_t, N_t, t) = G(K_t, X_t N_t)$, where X_t grows at a constant rate γ_X.
- Output, Y_t, investment, I_t, consumption, C_t, and capital, K_t grow at the same rate
$$\gamma_Y = \gamma_C = \gamma_I = \gamma_k = \gamma_\lambda.$$
- Household preferences, $u(c, l)$, have a particular form.

The theoretical reasoning is as follows.

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + I_t \\ \frac{K_{t+1}}{K_t} &= (1-\delta) + \frac{I_t}{K_t} \\ \frac{I_t}{K_t} &= \gamma_K - (1-\delta). \end{aligned}$$

Then $\gamma_I = \gamma_K$.

Proof. Suppose that $\gamma_I > \gamma_K$, then $\frac{I_t}{K_t} \to \infty$. Suppose that $\gamma_I < \gamma_K$, then $\frac{I_t}{K_t} \to 0$. □

Note that $Y_t = C_t + I_t$.

Proof. Suppose that $\gamma_C \neq \gamma_I$. If $\gamma_C > \gamma_I$, then $\frac{I_t}{C_t} \to 0$. Suppose that $\gamma_Y = \gamma_C$, so that $\frac{Y_t}{C_t} \to 1$ and $\frac{C_t}{Y_t} \to 1$. More generally, $\gamma_Y = \max\{\gamma_C, \gamma_I\}$ and $\frac{I_t}{Y_t} \to 0$. However, if you want $\frac{C_t}{Y_t} \neq 0$ and $\frac{I_t}{Y_t} \neq 0$, then it must be that $\gamma_Y = \gamma_C = \gamma_I$. Thus

$$\gamma_Y = \gamma_C = \gamma_I = \gamma_K,$$

and

$$\frac{Y_t}{K_t}, \frac{C_t}{K_t}, \frac{I_t}{K_t}, \frac{C_t}{Y_t}, \frac{I_t}{Y_t}$$

are constant over time. □

Theorem: Technology must be labor augmenting.

Proof. Take production function

$$Y_t = F(K_t, N_t, t).$$

Then

$$\frac{Y_t}{K_t} = F(1, \frac{N_t}{K_t}, t).$$

Constant

Grows at rate $\frac{\gamma_N}{\gamma_K}$

You can show that

$$F(1, \frac{N_t}{K_t}, t) = G(1, X_t \frac{N_t}{K_t}).$$

Define $G(K, N) = F(K, N, 0)$ for all K and N. Then by definition

$$G(1, \frac{N}{K}) = F(1, \frac{N}{K}, 0).$$

By constant returns to scale,

$$G(1, \frac{N}{K}) = \frac{1}{K} F(K, N, 0).$$

It follows that

$$G(1, \frac{N}{K}) = \frac{1}{K} G(K, N),$$

so $G(\cdot)$ has constant returns to scale. At time $t \geq 0$, then from constant returns to scale

$$\begin{aligned}
F(1, \tfrac{N}{K}, t) &= \tfrac{1}{K} F(K, N, t) \\
F(1, \tfrac{N}{K}, t) &= \tfrac{Y}{K} \\
\tfrac{Y_t}{K_t} &= \tfrac{Y_0}{K_0} \\
F(1, \tfrac{N}{K}, t) &= \tfrac{G(K_0, N_0)}{K_0} \\
F(1, \tfrac{N}{K}, t) &= G(1, \tfrac{N_0}{K_0}) \\
F(1, \tfrac{N}{K}, t) &= G(1, \tfrac{N_t}{K_t} (\tfrac{\gamma_N}{\gamma_K})^{-t}) \\
N_t &= N_0 \gamma_N^t \\
K_t &= K_0 \gamma_K^t \\
F(1, \tfrac{N}{K}, t) &= G(1, (\tfrac{\gamma_K}{\gamma_N})^t \tfrac{N_t}{K_t}).
\end{aligned}$$

Define $X_t \equiv (\frac{\gamma_K}{\gamma_N})^t$ and $\gamma_X \equiv \frac{\gamma_K}{\gamma_N}$. It follows that

$$
\begin{aligned}
F(K_t, N_t, t) &= K_t F(1, \tfrac{N_t}{K_t}, t) \\
F(K_t, N_t, t) &= K_t G(1, X_t \tfrac{N_t}{K_t} \\
F(K_t, N_t, t) &= G(K_t, X_t N_t) \\
G(K, N) &= F(K, N, 0).
\end{aligned}
$$

Note that $\frac{X_t N_t}{K_t}$ is constant and $\gamma_K = \gamma_X \gamma_N$. Also,

$$
\begin{aligned}
Y &= F(K, X_t N) \\
Y_t &= F(K_t, X_t N_t).
\end{aligned}
$$

Let $y \equiv \frac{Y}{XN}$ and $k \equiv \frac{K}{XN}$. Then by constant returns to scale

$$y = F(k, 1) = f(k).$$

The marginal product of capital is

$$
\begin{aligned}
\frac{\Delta F}{\Delta K} &= \frac{F(K + \Delta K, X_N) - F(K, X_n)}{\Delta K} \\
\frac{\Delta F}{\Delta K} &= \frac{X_N (F(\frac{K + \Delta K}{XN}, 1) - F(\frac{K}{XN}, 1))}{\Delta K} \\
\frac{\Delta F}{\Delta K} &= \frac{F(k + \frac{\Delta K}{XN}, 1) - F(k, 1)}{(\frac{\Delta K}{XN})} \\
\frac{\Delta F}{\Delta K} &= \frac{F(k + h, 1) - F(k, 1)}{h}.
\end{aligned}
$$

Note that

$$\frac{\partial f}{\partial k} = \frac{\partial F(k, 1)}{\partial k} = \frac{\mathrm{d} f}{\mathrm{d} k},$$

so

$$\mathrm{MPK} = \frac{\partial f(k, 1)}{\partial k} = f'(k),$$

and $\mathrm{MPK}_t = f'(k_t)$ is constant over time because $k_t = \frac{K_t}{X_t N_t}$. From Euler's identity and constant returns to scale,

$$
\begin{aligned}
Y_t &= \mathrm{MPN}_t N_t + \mathrm{MPK}_t K_t \\
Y_t &= w_t N_t + \mathrm{MPK}_t K_t \\
w_t N_t &= Y_t - \mathrm{MPK}_t K_t.
\end{aligned}
$$

So, the real wage is

$$w_t = \tfrac{Y_t}{N_t} - \mathrm{MPK}_t \tfrac{K_t}{N_t}$$

and grows at a constant rate. Furthermore,

$$\frac{\gamma_Y}{\gamma_N} = \frac{\gamma_K}{\gamma_N} = \gamma_X,$$

so

$$\gamma_Y = \gamma_C = \gamma_I = \gamma_K = \gamma_X \gamma_N.$$

Empirically, $\gamma_N \approx 1$. Thus

$$\gamma_Y = \gamma_C = \gamma_I = \gamma_K \approx \gamma_X.$$

$\square$

Example: King, Plosser, and Rebelo's Balanced Growth Model (1988)
The model begins with the program

$$
\begin{aligned}
&\max \sum_{t=0}^{T} \beta^t u(c,l) \\
&k_{t+1} = (1+r_t)k_t + w_t n_t - c_t \\
n_t + l_t &= 1
\end{aligned}
$$

The market clearing prices are

$$
\begin{aligned}
r_t &= \mathrm{MPK}_t - \delta = \frac{\partial f(K_t, X_t n_t)}{\partial K_t} - \delta \\
w_t &= \mathrm{MPN}_t = \frac{\partial f(K_t, X_t n_t}{\partial n_t}
\end{aligned}
$$

Balanced growth implies the following

- The growth rates
$$\gamma_y = \gamma_K = \gamma_C = \gamma_I = \gamma_X \gamma_N,$$
where $\gamma_N \approx 1$.
- The production function $F(K_t, N_t, t) = G(K_t, X_t n_t)$ has constant returns to scale.
- Preferences, $u(c,l)$, have a special functional form that are consistent with constant relative risk aversion (CRRA).

The Lagrangian for this model is

$$\mathscr{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) - \sum_{t=0}^{\infty} \lambda_t (k_{t+1} - (1+r_t)k_t - w_t n_t + c_t).$$

The first-order conditions are

$$
\begin{aligned}
\frac{\partial \mathscr{L}}{\partial c_t} &= \beta^t u_c(c_t, l_t) - \lambda_t = 0 \\
\frac{\partial \mathscr{L}}{\partial l_t} &= \beta^t u_l(c_t, l_t) - \lambda_t w_t = 0 \\
\frac{\partial \mathscr{L}}{\partial k_{t+1}} &= \lambda_t - \mathbb{E}_t(1+r_{t+1})\lambda_{t+1} = 0
\end{aligned}
$$

The intertemporal Euler equation is

$$u_c(c_t, l_t) = \beta\, \mathbb{E}_t(1+r_{t+1}) u_c(c_{t+1}, l_{t+1})$$

The intratemporal Euler equation is

$$\frac{u_l(c_t, l_t)}{u_c(c_t, l_t)} = w_t.$$

Note that the ratio of marginal utilities from consumption in two subsequent periods is constant

$$\frac{u_c(c_{t+1}, l_{t+1})}{u_c(c_t, l_t)} = \frac{1}{\beta(1+r_{t+1})} \equiv \xi.$$

So,

$$u_c(\gamma_c c_t, l_t) = \xi u_c(c_t, l_t)$$

for all c in a neighborhood of c_t. If you take the partial derivative with respect to consumption, then

$$\begin{aligned} \frac{\partial u_c(\gamma_c c_t, l_t)}{\partial c_t} &= \frac{\partial \xi u_c(c_t, l_t)}{\partial c_t} \\ \gamma_c u_{cc}(\gamma_c c_t, l_t) &= \xi u_{cc}(c_t, l_t). \end{aligned}$$

If you then impose a constant individual supply of labor l, then

$$u_c(c_{t+1}, l) = \xi u_c(c_t, l).$$

It follows from taking the ratio of the previous equations that

$$\begin{aligned} \frac{\gamma_c u_{cc}(c_{t+1}, l)}{u_c(c_{t+1}, l)} &= \frac{\xi u_{cc}(c_t, l)}{\xi u_c(c_t, l)} \\ \frac{c_{t+1} u_{cc}(c_{t+1}, l)}{u_c(c_{t+1}, l)} &= \frac{c_t u_{cc}(c_t, l)}{u_c(c_t, l)}, \end{aligned}$$

for all $t = 0, 1, \ldots$. So, the coefficient of relative risk aversion is constant

$$\frac{c_t u_{cc}(c_t, l)}{u_c(c_t, l)} = -\sigma_c.$$

If you now solve for u_c, then

$$\begin{aligned} \frac{\mathrm{d} \log u_c(c_t, l)}{\mathrm{d} c_t} &= -\sigma_c \\ \mathrm{d} \log u_c(c_t, l) &= -\sigma_c \frac{\mathrm{d} c_t}{c_t}. \end{aligned}$$

If you integrate both sides, then

$$\begin{aligned} \log u_c(c_t, l) &= -\sigma_c \log(c_t) + v(l) \\ u_c(c_t, l) &= \mathrm{e}^{-\sigma_c \log(c_t)} \mathrm{e}^{v(l)} \\ u_c(c_t, l) &= c_t^{-\sigma_c} \mathrm{e}^{v(l)}. \end{aligned}$$

If you now solve for u, then

$$\begin{aligned} \frac{\mathrm{d} u(c_t, l)}{\mathrm{d} c_t} &= c_t^{-\sigma_c} \mathrm{e}^{v(l)} \\ \mathrm{d} u(c_t, l) &= c_t^{-\sigma_c} \mathrm{e}^{v(l)} \, \mathrm{d} c_t. \end{aligned}$$

If you now integrate both sides, then you find that there is a constant relative risk aversion (CRRA) utility function

$$u(c_t, l) = \begin{cases} \frac{c_t^{1-\sigma_c}}{1-\sigma_c} \mathrm{e}^{v_1(l)} + v_2(l) & \text{if } \sigma_c \neq 1, \\ \log(c_t) \mathrm{e}^{v_1(l)} + v_2(l) & \text{if } \sigma_c = 1. \end{cases}$$

Theorem: CRRA preferences satisfy the conditions for balanced growth.

Proof. Recall that the intratemporal Euler equation is

$$\frac{u_l(c_t, l_t)}{u_c(c_t, l_t)} = w_t.$$

With CRRA preferences, then

$$\begin{aligned} u_l(c_t, l_t) &= \frac{c_t^{1-\sigma_c}}{1-\sigma_c} v_1'(l) + v_2'(l) \\ u_c(c_t, l_t) &= c_t^{-\sigma_c} v_1(l). \end{aligned}$$

It follows that

$$\underbrace{\frac{c_t}{1-\sigma_c}}_{\text{Grows at } \gamma_c} \underbrace{\frac{v_1'(l)}{v_1(l)}}_{\text{Constant}} + \underbrace{c_t^{\sigma}}_{\text{Grows at } \gamma_c^{\sigma}} \underbrace{\frac{v_2'(l)}{v_1(l)}}_{\text{Constant}} = \underbrace{w_t}_{\text{Grows at } \gamma_w = \gamma_c}.$$

In order for the above equation to hold on the balanced growth path, then it must be that

$$v_2'(l) = 0.$$

This implies that $v_2(l)$ is constant. So, preferences must be

$$u(c_t, l_t) = \frac{c_t^{1-\sigma_c}}{1-\sigma_c} v_1(l).$$

If $\sigma_c = 1$, then $v_1(l)$ is constant and $v_2(l)$ is unrestricted

$$u(c_t, l_t) = X \log(c_t) + v_2(l).$$

□

Also, note that a linear utility function does not satisfy the balanced growth path[3].

[3] The linear utility function

$$u(c_t, l_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + \frac{(1-l_t)^{1-X}}{1-X}$$

assumes that growth is constant. This abstracts from growth and does not satisfy the conditions of the balanced growth path.

3.4.2. Real Business Cycles

The following is a rough guide as to how to solve real business cycle (RBC) models. You can extend the real business cycle model with fiscal policy shocks, monetary policy shocks, investment–specific shocks, financial frictions, risk–premiums, open–economies, and much more.

Consider an economy with a Cobb–Douglas production function and where the representative agent has a CRRA utility function

$$
\begin{aligned}
Y_t &= Z_t K_t^{1-\alpha}(X_t N_t)^{\alpha} \\
u(c_t, l_t) &= \frac{c_t^{1-\sigma}}{1-\sigma}\frac{l_t^{1-x}}{1-x}.
\end{aligned}
$$

1. The first step is to write the Lagrangian of the representative household

$$
\mathscr{L} = \sum_{t=0}^{\infty} \beta^t \Big[u(c_t, l_t) + \lambda(k_{t+1} - (1+r_t)k_t - w_t n_t + c_t) \Big].
$$

   The first–order conditions are

$$
\begin{aligned}
c_t^{-\sigma}\frac{l_t^{1-x}}{1-x} &= \lambda_t \\
\frac{c_t}{1-\sigma}\frac{l_t}{1-x} &= w_t \\
\beta(1+r_{t+1})\lambda_{t+1} &= \lambda_t \\
(1+r_{t+1})k_t + w_t n_t - c_t &= k_{t+1}.
\end{aligned}
$$

   You can then find the real wage in the economy

$$
\begin{aligned}
w_t &= \alpha Z_t K_t^{1-\alpha}(X_t N_t)^{\alpha-1} \\
w_t &= \alpha \frac{Y_t}{N_t}.
\end{aligned}
$$

   The marginal product of capital in the economy is

$$
\begin{aligned}
\text{MPK}_t &= (1-\alpha) Z_t K_t^{-\alpha}(X_t N_t)^{\alpha} \\
\text{MPK}_t &= (1-\alpha)\frac{Y_t}{K_t}.
\end{aligned}
$$

   So, the real rental rate of capital is

$$
r_t = \text{MPK}_t - \delta.
$$

2. The next step is to solve for the nonstochastic steady–state. Note that the model as is has a balanced growth path, so, to find a steady–state then you must transform the model's variables. Define

$$
\widetilde{y}_t = \frac{Y_t}{X_t},\ \widetilde{c}_t = \frac{C_t}{X_t},\ \widetilde{i}_t = \frac{I_t}{X_t},\ \widetilde{k}_t = \frac{K_t}{X_t}.
$$

   You can then rewrite the first–order conditions. The normalized wage rate is

$$
\widetilde{w}_t = \frac{\widetilde{c}_t}{1-\sigma}\frac{l_t}{1-x}.
$$

For the shadow price of capital

$$\lambda_t = c_t^{-\sigma} \frac{l_t^{1-x}}{1-x}$$
$$\frac{\lambda_t}{X_t^{-\sigma}} = \left(\frac{c_t}{X_t}\right)^{-\sigma} \frac{l_t^{1-x}}{1-x}$$
$$X_t^{\sigma}\lambda_t = \widetilde{c}_t^{-\sigma} \frac{l_t^{1-x}}{1-x}$$
$$\widetilde{\lambda}_t = \widetilde{c}_t^{-\sigma} \frac{l_t^{1-x}}{1-x},$$

where $\widetilde{\lambda}_t \equiv X_t^{\sigma}\lambda_t$. Also,

$$\lambda_t = \beta(1+r_{t+1})\lambda_{t+1}$$
$$X_t^{\sigma}\lambda_t = \beta(1+r_{t+1})X_t^{\sigma}\lambda_{t+1}$$
$$\widetilde{\lambda}_t = \beta(1+r_{t+1})\frac{X_t^{\sigma}}{X_{t+1}^{\sigma}}X_{t+1}^{\sigma}\lambda_{t+1}$$
$$\widetilde{\lambda}_t = \beta(1+r_{t+1})\gamma_X^{-\sigma}\widetilde{\lambda}_{t+1}.$$

Note that $\widetilde{w}$, $\widetilde{k}$, $\widetilde{y}$, and $\widetilde{\lambda}$ are constant

$$\widetilde{w}_t = \widetilde{w}_{t+1} = \widetilde{w}$$
$$\widetilde{k}_t = \widetilde{k}_{t+1} = \widetilde{k}$$
$$\widetilde{y}_t = \widetilde{y}_{t+1} = \widetilde{y}$$
$$\widetilde{\lambda}_t = \widetilde{\lambda}_{t+1} = \widetilde{\lambda}.$$

Notice that you can then find the steady–state real rate of return as a function of exogenous parameters

$$\beta(1+r)\gamma_X^{-\sigma}\widetilde{\lambda} = \widetilde{\lambda}$$
$$\widetilde{\lambda}(1-\beta(1+r)\gamma_X^{-\sigma}) = 0$$
$$\beta(1+r)\gamma_X^{-\sigma} = 1$$
$$1+r = \tfrac{\gamma_X^{\sigma}}{\beta}.$$

You can then use the motion of capital $k_{t+1} = (1-\delta)k_t + y_t - c_t$ to solve for k. In general, you can solve $\widetilde{y}$, $\widetilde{c}$, $\widetilde{i}$, $\widetilde{k}$, $\widetilde{w}$, $\widetilde{\lambda}$, r, n, etc.

3. The next step is to log–linearize each transformed equation around the nonstochastic steady–state. A brief algorithm for log–linearizing an expression is given below. Suppose you have a function, $f(x_t)$, and a deterministic steady–state, $\overline{x}$. You can obtain the linear approximation around the steady–state

$$f(x_t) \approx f(\overline{x}) + f'(x_t)(x_t - \overline{x}).$$

If you then take the natural logarithm of both sides and rearrange, then

$$\log f(x_t) \approx \log f(\overline{x}) + \log f'(x_t)(x_t - \overline{x})$$
$$\log f(x_t) - \log f(\overline{x}) \approx \log f'(x_t)(x_t - \overline{x}).$$

Note that the difference in logs is approximately interpreted as a growth rate

$$\log(x_t) - \log(\overline{x}) \approx \frac{x_t - \overline{x}}{\overline{x}}.$$

Example: Log–Linearization
Consider a function, $f(A_t) = \alpha A_t$, with a steady–state, A. The log–linear approximation around the steady–state can be found

$$\begin{aligned}
f(A_t) &\approx f(A) + f'(A_t)(A_t - A)\\
\log f(A_t) &\approx \log f(A) + \log f'(A_t)(A_t - A)\\
\log \alpha A_t &\approx \log \alpha A + \frac{\partial \log(\alpha A_t)}{\partial \log A_t}(\log A_t - \log A)\\
\log \alpha A_t &\approx \log \alpha A + \frac{\partial(\log \alpha + \log A_t)}{\partial \log A_t}(\log A_t - \log A)\\
\log \alpha A_t &\approx \log \alpha A + (1)(\log A_t - \log A)\\
\log \alpha A_t - \log \alpha A &\approx \log A_t - \log A \equiv \hat{A}_t.
\end{aligned}$$

Example: Log–Linearization
Consider a function, $f(A_t, B_t) = \frac{A_t}{B_t}$, with a steady–states values $\overline{A}$ and $\overline{B}$. The log–linear approximation around the steady–state can be found

$$\begin{aligned}
\log\left(\frac{A_t}{B_t}\right) &\approx \log\left(\frac{\overline{A}}{\overline{B}}\right) + \begin{pmatrix} \partial[\log A_t - \log B_t]/\partial A_t \\ \partial[\log A_t - \log B_t]/\partial B_t \end{pmatrix}' \begin{pmatrix} \log A_t - \log \overline{A} \\ \log B_t - \log \overline{B} \end{pmatrix}\\
\log\left(\frac{A_t}{B_t}\right) &\approx \log\left(\frac{\overline{A}}{\overline{B}}\right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} \log A_t - \log \overline{A} \\ \log B_t - \log \overline{B} \end{pmatrix}\\
\log\left(\frac{A_t}{B_t}\right) - \log\left(\frac{\overline{A}}{\overline{B}}\right) &\approx (\log A_t - \log \overline{A}) - (\log B_t - \log \overline{B})\\
\left(\frac{\hat{A}_t}{B_t}\right) &\approx \hat{A}_t - \hat{B}_t.
\end{aligned}$$

Continuing with the transformed Euler equations

$$\begin{aligned}
\widetilde{\lambda}_t &= \widetilde{c}_t^{-\sigma} \tfrac{l_t^{1-x}}{1-x}\\
\widetilde{w}_t &= \tfrac{\widetilde{c}_t}{1-\sigma} \tfrac{l_t}{1-x}\\
\widetilde{\lambda}_t &= \beta(1 + r_{t+1})\gamma_x^{-\sigma}\widetilde{\lambda}_{t+1},
\end{aligned}$$

the equations can be log-linearized around their steady–state values

$$\hat{\widetilde{\lambda}}_t = -\sigma\hat{\widetilde{c}}_t + (1-x)\hat{l}_t \tag{1}$$

$$\hat{\widetilde{w}}_t = \hat{\widetilde{c}}_t + \hat{\widetilde{l}}_t \tag{2}$$

$$\hat{\widetilde{\lambda}}_t = (1 \hat{+} r_{t+1}) + \hat{\widetilde{\lambda}}_{t+1}. \tag{3}$$

4. The final steps are to put the system into Blanchard–Kahn form and solve. This is an application of solving a system of forward–looking linear equations. You can use equation (1) to substitute for $\hat{\widetilde{c}}_t$ everywhere. Next, you can use equation (2) to substitute for $\hat{\widetilde{l}}_t$ everywhere. Next, you can log–linearize the time constraint

$$\begin{aligned}
l_t + n_t &= 1\\
\frac{l_t}{l_t + n_t}\hat{l}_t + \frac{n_t}{l_t + n_t}\hat{n}_t &= 0\\
l_t\hat{l}_t + n_t\hat{n}_t &= 0,
\end{aligned}$$

and substitute for $\hat{n}_t$ everywhere. You will be left with 2 equations with variables $\hat{\hat{k}}_t$, $\hat{\hat{k}}_{t+1}$, $\hat{\hat{\lambda}}_t$, $\hat{\hat{\lambda}}_{t+1}$, and the exogenous shock z_t. The initial capital, k_0, is given. You can then search for an unique non–explosive equilibrium. You can put the system into Blanchard–Kahn form

$$\begin{pmatrix} \hat{\hat{k}}_{t+1} \\ \hat{\hat{\lambda}}_{t+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{\hat{k}}_t \\ \hat{\hat{\lambda}}_t \end{pmatrix} + \mathbf{Q}\hat{z}_t.$$

You can then solve for $\hat{\hat{\lambda}}_t$ as a function of $\hat{\hat{k}}_t$ and $\hat{\hat{z}}_t$. You can also solve for $\hat{\hat{c}}_t$, $\hat{\hat{l}}_t$, $\hat{\hat{n}}_t$, $\hat{\hat{y}}_t$, $\hat{\hat{i}}_t$, and $(1 \hat{+} r)_{t+1}$ as functions of $\hat{\hat{k}}_t$ and $\hat{z}_t$. This solves the linearized version of the real business cycle model. The linear approximation is valid close to the steady–state. If you have persistent shocks to $\hat{z}_t$, such as an AR(1) process

$$\hat{z}_t = \rho \hat{z}_{t-1} + \varepsilon_t,$$

then you can treat $\hat{z}_{t-1}$ as an additional state variable with exogenous shock ε_t. The Blanchard–Kahn form is then

$$\begin{pmatrix} \hat{z}_t \\ \hat{\hat{k}}_{t+1} \\ \hat{\hat{\lambda}}_{t+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{z}_{t-1} \\ \hat{\hat{k}}_t \\ \hat{\hat{\lambda}}_t \end{pmatrix} + \mathbf{Q}\varepsilon_t.$$

You can then do Blanchard–Kahn in two–dimensional form, but use

$$\mathbb{E}_t\, \hat{z}_{t+j} = \rho^j \hat{z}_t,$$

for all j. You can then solve for $\hat{\hat{\lambda}}_t$ accordingly. Note that the Blanchard–Kahn solution looks similar to a VAR. With the log–linearized real business cycle model above, you have

$$\begin{aligned} \hat{\hat{c}}_t &= \left(1 - \frac{\gamma+\delta}{1+r_{t+1}}\frac{1}{1+\sigma}\right)\mathbb{E}_t\, \hat{\hat{c}}_{t+1} + \frac{\gamma+\delta}{1+r_{t+1}}\kappa_1\, \mathbb{E}_t\, \hat{X}_{t+1} \\ \hat{\hat{k}}_{t+1} &= \kappa_2 \hat{\hat{k}}_t + \kappa_3 \hat{\hat{c}}_t - \kappa_4 \hat{z}_t. \end{aligned}$$

The system can be written in Blanchard–Kahn form

$$\begin{pmatrix} \hat{\hat{k}}_{t+1} \\ \hat{\hat{c}}_{t+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{\hat{k}}_t \\ \hat{\hat{c}}_t \end{pmatrix} + \mathbf{Q}\hat{z}_t,$$

where $\hat{\hat{k}}_{t+1}$ is the sate variable and $\hat{\hat{c}}_{t+1}$ is the jump variable. Define the system as

$$x_{t+1} = \mathbf{A}x_t + \mathbf{Q}z_t.$$

You can diagonalize the matrix $\mathbf{A}$,

$$\mathbf{A} = P\Lambda P^{-1}.$$

Next you can substitute and normalize the system

$$\begin{aligned} x_{t+1} &= P\Lambda P^{-1}x_t + \mathbf{Q}z_t \\ P^{-1}x_{t+1} &= \Lambda P^{-1}x_t + P^{-1}\mathbf{Q}z_t \\ \hat{x}_{t+1} &= \Lambda \hat{x}_t + \mathbf{Q}\hat{z}_t. \end{aligned}$$

Example: The Blanchard–Kahn Method
You can use the Blanchard–Kahn method to solve the system

$$\mathbb{E}\begin{bmatrix} k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = \mathbf{A}\begin{bmatrix} k_t \\ \lambda_t \end{bmatrix} + \mathbf{Q}\varepsilon_t.$$

First, you can find that one equation has an explosive root by diagonalize $\mathbf{A}$

$$\mathbf{A} = P\Lambda P^{-1},$$

where

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}.$$

Without loss of generality

$$|\Lambda_2| > 1.$$

Note that $\mathbb{E}_t\,\lambda_{t+1}$ is unstable and λ_t is a jump variable. Next, you can substitute for $\mathbf{A}$, divide by P, and perform a change of variables

$$\begin{aligned} \mathbb{E}\begin{bmatrix} k_{t+1} \\ \lambda_{t+1} \end{bmatrix} &= P\Lambda P^{-1}\begin{bmatrix} k_t \\ \lambda_t \end{bmatrix} + \mathbf{Q}\varepsilon_t \\ P^{-1}\,\mathbb{E}\begin{bmatrix} k_{t+1} \\ \lambda_{t+1} \end{bmatrix} &= \Lambda P^{-1}\begin{bmatrix} k_t \\ \lambda_t \end{bmatrix} + P^{-1}\mathbf{Q}\varepsilon_t \\ \mathbb{E}\,P^{-1}z_{t+1} &= \Lambda P^{-1}z_t + P^{-1}\mathbf{Q}\varepsilon_t. \end{aligned}$$

If you impose a condition that the soulition should look like

$$\mathbb{E}_t\,\widetilde{z}_{t+1} = \Lambda\widetilde{z}_t + P^{-1}\mathbf{Q}\varepsilon_t,$$

then

$$\mathbb{E}_t\,\lambda_{t+j} = \Lambda_2^j\lambda_t = 0.$$

You can now solve for a solution to the system.

3.5. Monopolistic Competition and Nominal Rigidities

3.5.1. Nominal Rigidities

Reading: Mankiw, N. Gregory (1985). "Small Menu Costs and Large Business Cycles: A Macroeconomic Model of Monopoly," *Quarterly Journal of Economics* 100, 529-537.

Consider if there are costs for adjusting prices.

Definition: Menu Costs
Menu costs are small direct costs, with larger indirect cots, for adjusting prices.

Even second–order menu costs can have large effects on prices, quantities and welfare. It is often not worthwhile for a firm to change its price.

Example: Suppose that there is a firm with price p. Assume that there is a menu cost, z, to change price. The demand elasticity of the firm is θ such that

$$\frac{\mathrm{d}\log Q}{\mathrm{d}\log(p/P)} = -\theta.$$

Changes in P have the same effect in Q as changes in p. Note that

$$\mathrm{d}\log(p/p) = \mathrm{d}\log p - \mathrm{d}\log P.$$

So, first–order effects of changes in P on Q is

$$\begin{aligned}
\mathrm{d}\log Q &= -\theta(\mathrm{d}\log p - \mathrm{d}\log P) \\
\mathrm{d}\log Q &= \theta\,\mathrm{d}\log P \\
\frac{\mathrm{d}\log Q}{\mathrm{d}\log P} &= \theta,
\end{aligned}$$

but the effects of P on the firm's profits are second–order

$$\frac{\mathrm{d}\pi}{\mathrm{d}(p/P)} = 0.$$

If there are second–order menu costs, z, the firm may not change its price, p. Therefore, the menu costs, z, have fist–order effects on output, Q.

Example: Suppose that there is a monopolist that posts price p in advance, and changing price incurs cost c. Also, assume that there are shocks to the aggregate price index P. Let the shocks be $\mathrm{d}P$ and the effects be $\mathrm{d}Q$. The slope of the monopolist's demand curve is $\frac{\mathrm{d}p}{\mathrm{d}Q}$. The total change in quantity, $\mathrm{d}Q$, is then the total change in price, $\mathrm{d}P$, divided by the slope of the monopolist's demand curve. Note that, because of the monopolist's first–order conditions,

$$\frac{\partial\pi}{\partial P} = 0.$$

Thus, for small changes in P it is not worthwhile to change prices. The key result is that nominal rigidities require monopolistic competition.

3.5.2. Monopolistic Competition

Reading: Avinash K. Dixit and Joseph E. Stiglitz (1977). "Monopolistic Competition and Optimum Product Diversity," *American Economic Review* 67, 297-308.

Reading: Blanchard, Olivier, and Nobuhiro Kiyotaki (1987). "Monopolistic Competition and the Effects of Aggregate Demand," *American Economic Review* 77, 647-666.

Dixit and Stiglitz introduce a static, one–period, monopolistic competition macroeconomic model with microfoundations. There are M varieties of goods and money, Y, in the economy. The representative household has CES preferences

$$U(c_1, \ldots, c_m, Y, N) = \left(\sum_{i=1}^{M} c_i^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}\gamma} \left(\frac{Y}{P}\right)^{1-\gamma} - N^{\beta},$$

where $\gamma \in (0,1)$, $\beta \geq 1$, and $\epsilon > 0$. Let p_i denote the price of good i. The budget constraint of the household is

$$\sum_{i=1}^{M} p_i c_i + Y = wN.$$

The Lagrangian can be written as

$$\mathscr{L} = \left(\sum_{i=1}^{M} c_i^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}\gamma} \left(\frac{Y}{P}\right)^{1-\gamma} - N^{\beta} + \lambda(wN - PY - \sum_{i=1}^{M} p_i c_i).$$

The first order conditions for two goods, i and j, are

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial c_i} &= \left(\sum_{i=1}^{M} c_i^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}\gamma-1} \left(\frac{\epsilon}{\epsilon-1}\right)\left(\frac{\epsilon-1}{\epsilon}\right) c_i^{\frac{\epsilon-1}{\epsilon}-1} \left(\frac{Y}{P}\right)^{1-\gamma} - \lambda p_i = 0 \\
\frac{\partial \mathscr{L}}{\partial c_j} &= \left(\sum_{i=1}^{M} c_i^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}\gamma-1} \left(\frac{\epsilon}{\epsilon-1}\right)\left(\frac{\epsilon-1}{\epsilon}\right) c_j^{\frac{\epsilon-1}{\epsilon}-1} \left(\frac{Y}{P}\right)^{1-\gamma} - \lambda p_j = 0.
\end{aligned}$$

It follows that for all goods j, that

$$\begin{aligned}
\frac{c_i^{\frac{\epsilon-1}{\epsilon}-1}}{c_j^{\frac{\epsilon-1}{\epsilon}-1}} &= \frac{p_i}{p_j} \\
\frac{c_i^{\frac{\epsilon-1}{\epsilon}-\frac{\epsilon}{\epsilon}}}{c_j^{\frac{\epsilon-1}{\epsilon}-\frac{\epsilon}{\epsilon}}} &= \frac{p_i}{p_j} \\
\frac{c_i^{-\frac{1}{\epsilon}}}{c_j^{-\frac{1}{\epsilon}}} &= \frac{p_i}{p_j} \\
\left(\frac{c_i}{c_j}\right)^{-\frac{1}{\epsilon}} &= \frac{p_i}{p_j} \\
\frac{c_i}{c_j} &= \left(\frac{p_i}{p_j}\right)^{-\epsilon}.
\end{aligned}$$

The household's optimality condition is

$$\frac{c_i}{c_j} = \left(\frac{p_i}{p_j}\right)^{-\epsilon},$$

and the elasticity of substitution is

$$\frac{\mathrm{d}\log(c_i/c_j)}{\mathrm{d}\log(p_i/p_j)} = -\epsilon.$$

Define

$$C \equiv \left(\sum_{i=1}^{M} c_i^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}$$

$$P \equiv \left(\sum_{i=1}^{M} p_i^{1-\epsilon}\right)^{\frac{1}{1-\epsilon}}$$

It follows that

$$c_i = \left(\frac{p_i}{P}\right)^{-\epsilon} C,$$

and

$$\sum_{i=1}^{M} p_i c_i = PC.$$

The households view items as identical. Preferences aggregate across goods. You can write the household's utility as

$$U(C, Y, N) = C^{\gamma}\left(\frac{Y}{P}\right)^{1-\gamma} - N^{\beta},$$

where the utility function is additively separable in labor with Cobb–Douglas utility over the combination of goods, C, and money, Y. From the budget constraint, you can find that a household's supply of labor is

$$N = \frac{PC + Y}{w}.$$

The household's consumption of goods can be found from the optimization problem

$$\max_{C,\ Y} U = C^{\gamma} Y^{1-\gamma} - \left(\frac{PC + Y}{w}\right)^{\beta}.$$

The first order conditions are

$$\gamma C^{\gamma-1} Y^{1-\gamma} - \left(\frac{PC + Y}{w}\right)^{\beta-1} \beta P = 0$$

$$(1-\gamma) C^{\gamma} Y^{-\gamma} - \left(\frac{PC + Y}{w}\right)^{\beta-1} \beta = 0.$$

It follows that

$$\frac{\gamma}{1-\gamma}\frac{Y}{C} = P,$$

and the consumption of household h is

$$C_h^{*} = \frac{\gamma}{1-\gamma}\left(\frac{Y}{P}\right).$$

Next, the firms set prices where each household, $j = 1, \ldots, n$, has a downward–sloping demand for good i

$$c_{h,i} = \left(\frac{p_i}{P}\right)^{-\epsilon} C_h.$$

The aggregate demand for good i is

$$\begin{aligned}
\mathbb{C}_i &= \sum_{h=1}^{H} c_{h,i} \\
\mathbb{C}_i &= \left(\frac{p_i}{P}\right)^{-\epsilon} \sum_{h=1}^{H} C_h.
\end{aligned}$$

Thus the demand curve for firm i is

$$\mathbb{C}_i = \left(\frac{p_i}{P}\right)^{-\epsilon} \mathbb{C},$$

where $\mathbb{C}$ is aggregate consumption. If the variety of goods, M, is very large, then firm i takes aggregate consumption, $\mathbb{C}$, and the price index, P, as given. Consider firm i's profit maximization problem

$$\begin{aligned}
&\max_{p_i, Y_i, N_i} \pi = p_i Y_i - w N_i \\
&\text{s. t. } Y_i = N_i^{\alpha} \\
&\qquad Y_i = \left(\frac{p_i}{P}\right)^{-\epsilon} Y,
\end{aligned}$$

where $\alpha \in (0,1)$. By substitution, the profit maximization problem is

$$\begin{aligned}
&\max_{p_i, Y_i} \pi = p_i Y_i - w Y_i^{\frac{1}{\alpha}} \\
&\max_{p_i} \pi = p_i \left(\frac{p_i}{P}\right)^{-\epsilon} Y - w \left(\frac{p_i}{P}\right)^{-\frac{\epsilon}{\alpha}} Y^{\frac{1}{\alpha}} \\
&\max_{p_i} \pi = p_i^{1-\epsilon} \left(\frac{1}{P}\right)^{-\epsilon} Y - w p_i^{-\frac{\epsilon}{\alpha}} \left(\frac{1}{P}\right)^{-\frac{\epsilon}{\alpha}} Y^{\frac{1}{\alpha}}
\end{aligned}$$

The first–order condition is

$$\begin{aligned}
(1-\epsilon) p_i^{-\epsilon} \left(\frac{1}{P}\right)^{-\epsilon} Y + w \left(\frac{\epsilon}{\alpha}\right) p_i^{-\frac{\epsilon}{\alpha}-1} \left(\frac{1}{P}\right)^{-\frac{\epsilon}{\alpha}} Y^{\frac{1}{\alpha}} &= 0 \\
(1-\epsilon) \left(\frac{p_i}{P}\right)^{-\epsilon} Y + \left(\frac{w}{P}\right) \left(\frac{\epsilon}{\alpha}\right) \left(\frac{p_i}{P}\right)^{-\frac{\epsilon}{\alpha}-1} Y^{\frac{1}{\alpha}} &= 0 \\
(1-\epsilon) \left(\frac{p_i}{P}\right)^{-\epsilon} + \left(\frac{w}{P}\right) \left(\frac{\epsilon}{\alpha}\right) \left(\frac{p_i}{P}\right)^{-\frac{\epsilon}{\alpha}-1} Y^{\frac{1}{\alpha}-1} &= 0 \\
(1-\epsilon) + \left(\frac{w}{P}\right) \left(\frac{\epsilon}{\alpha}\right) \left(\frac{p_i}{P}\right)^{-\frac{\epsilon}{\alpha}-1+\epsilon} Y^{\frac{1}{\alpha}-1} &= 0
\end{aligned}$$

It follows that

$$\begin{aligned}
\left(\frac{w}{P}\right)\left(\frac{p_i}{P}\right)^{-\frac{\epsilon}{\alpha}-1+\epsilon} Y^{\frac{1}{\alpha}-1} &= \frac{\epsilon-1}{\epsilon}\alpha \\
\left(\frac{p_i}{P}\right)^{-\frac{\epsilon}{\alpha}-1+\epsilon} &= \frac{\epsilon-1}{\epsilon}\alpha\left(\frac{w}{P}\right)^{-1} Y^{1-\frac{1}{\alpha}}.
\end{aligned}$$

Thus

$$\left(\frac{p_i}{P}\right)^{1+\epsilon\left(\frac{1}{\alpha}-1\right)} = \frac{\epsilon}{\epsilon-1}\left(\frac{1}{\alpha}\frac{w}{P}Y^{\frac{1}{\alpha}-1}\right).$$

Markup above marginal cost   Marginal cost

Note that if the firm's demand is inelastic, then the firm would always set an infinite price. Thus, the only nonexplosive equilibrium is for the demand for each good to be elastic. All firms have the same production technology and face the same demand. This implies that each firm sets the same optimal price, $p_i = p_{-i}$ for all i. Thus,

$$\frac{p_i}{P} = 1.$$

Furthermore, there is a monopolistic underprovision externality, because each firm is maximizing profits seperately there is a coordinationo failure to increase production. So,

$$\text{MPL} > \frac{w}{P} = \text{MRS}_h.$$

Suppose that prices are set in advance before money supply, Y. Then the central bank can influence aggregate consumption, C, through changes in the real money balance, $\frac{Y}{P}$. Note that everyone in the economy could be better off, even if firms could only change p_i by paying a menu cost. However, first–order effects of money supply, Y, and aggregate consumption, C, only induce second–order effects on firm profits and a firm i may not change p_i. The conclusion is that the introduction of monopolistic competition may lead to sticky prices in an economy.

Example: Symmetric Monopolistic Competition in the Labor Market
Suppose that there are n different types of labor and that each household, $j = 1, \ldots, n$, is a monopoly supplier of labor type j. The production function of firm i is

$$Y_i = \left(\sum_{j=1}^{n} N_j^{\frac{\sigma-1}{\sigma}}\right)^{\alpha\frac{\sigma}{\sigma-1}}.$$

Firm i's demand for labor j is

$$N_{ij} = \left(\frac{w_j}{w}\right)^{-\sigma} N_i,$$

where

$$\begin{aligned}
N_i &\equiv \left(\sum_{j=1}^{n} N_{ij}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \\
w &\equiv \left(\sum_{j=1}^{n} w_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}.
\end{aligned}$$

It is true that

$$\sum_{j=1}^{n} w_j N_{ij} = wN_i.$$

The profits of firm i are

$$\pi_i = p_i Y_i - \sum_{j=1}^{n} w_j N_{ij}$$
$$\pi_i = p_i Y_i - wN_i,$$

where $Y_i = N_i^{\alpha}$. Thus, the firm's optimal choice of labor, N_i, is the same as in the case with homogeneous labor. However, the wage, w, is higher than the competitive wage.

Each household faces demand for labor

$$N_{ij} = \left(\frac{w_i}{w}\right)^{-\sigma} N_i,$$

from each firm $i = 1, \ldots, m$. The total demand for the labor of household j is

$$\sum_{i=1}^{m} N_{ij} = \left(\frac{w_j}{w}\right)^{-\sigma} \sum_{i=1}^{m} N_i$$
$$\mathbb{N}_j = \left(\frac{w_j}{w}\right)^{-\sigma} \mathbb{N},$$

where $\mathbb{N}$ is the aggregate demand for labor. Household j sets w_j to maximize their utility subject to the demand curve. The resulting first–order condition is

$$\left(\frac{w_j}{w}\right)^{1+\sigma(\beta-1)} = \frac{\sigma}{\sigma - 1}\kappa_w \frac{P}{w} Y^{\frac{1}{\alpha}(\beta-1)}.$$

Note that the households are undersupplying labor. Thus, a higher degree of monopolistic competition in the labor markets increases the monopolistic underproduction externality.

3.6. New Keynesian Economics

Classical macroeconomics assumes full employment, aggregate supply equals aggregate demand, general equilibrium, and perfectly flexible prices and wages. Keynesian economics assumes that the economy is in disequilibrium, nominal rigidities, aggregate supply does not equal aggregate demand, and there is unemployment. New classical macroeconomics assumes full employment, frictional unemployment, dynamic general equilibrium, aggregate supply equals aggregate demand, and frictionless adjustment of prices and wages. New Keynesian economics provides a framework to incorporate nominal rigidities, menu–costs, contracts, dynamic general equilibrium, and the aggregate supply equals aggregate demand while there are still imbalances between the trend of supply and demand. New Keynesian models begin with microfoundations and are built with representative households, monopolistic firms, and sticky prices, such as menu–costs, lagged price setting, and staggered contracts.

The New Keynesian DSGE model produces monetary policy effects from unexpected shocks to the nominal interest rate, $\varepsilon_{i,t} \neq 0$. The New Keynesian DSGE model is a structural model, so, you can use the model to study monetary policy. For possible extensions, you can add: a probability that firms use rule–of–thumb pricing, fiscal policy (i.e. government spending G_t), endogenous capital K_t by including investment such that $Y_t = C_t + I_t + G_t$ (however this model is volatile without including capital adjustment costs), as well as sticky wages. Adding sticky wages to the model results in unemployment and captures the true essence of Keynesian economics.

3.6.1. The New Keynesian DSGE Model

The New Keynesian Dynamic Stochastic General Equilibrium (NK–DSGE) considers an economy with monopolistic firms and competitive labor. The model is used to determine optimal policy decisions in response to exogenous shocks that cause deviation from the economy's steady–state. There is no capital for simplicity, the justification is that k_t is smooth in the short–run. Household consumption and price of goods from all firms are

$$
\begin{aligned}
C_t &= \left(\int_0^1 c_t(i)^{(\epsilon-1)/\epsilon} \, \mathrm{d}i \right)^{\epsilon/(\epsilon-1)} \\
P_t &= \left(\int_0^1 p_t(i)^{(\epsilon-1)/\epsilon} \, \mathrm{d}i \right)^{\epsilon/(\epsilon-1)}.
\end{aligned}
$$

The household's problem is

$$
\max_{\{C_t,N_t\}_{t=0}^{\infty}} U = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{N_t^{1-\varphi}}{1-\varphi} \right] Z_t
$$

$$
\text{s.t.} \quad P_t C_t + Q_t B_t = B_{t-1} + w_t N_t + D_t,
$$

where B_t is the household's stock of bonds that pay 1 unit of consumption, Q_t is the price of a 1 period nominal discounted bond (typically $Q_t < 1$, D_t are dividends, N_t is the household's labor supply to all firms, and w_t is the wage rate in period t. Notice that if you let

$$
Q = \frac{1}{1+i_t},
$$

with interest rate i_t, then the problem is identical to the standard RBC model. The budget constraint can be written as

$$
B_t = (1+i_t)(B_{t-1} + w_t N_t + D_t - P_t C_t).
$$

The stochastic exogenous shocks

$$z_t = \log Z_t$$

follow an AR(1) process

$$z_t = \rho_z + \epsilon_{z,t}.$$

The Lagrangian for the household's problem is

$$\max_{\{C_t,N_t\}_{t=0}^{\infty}} \mathscr{L} = \sum_{t=0}^{\infty} \beta^t \Big[\frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{N_t^{1-\varphi}}{1-\varphi}\Big] Z_t - \sum_{t=0}^{\infty} \lambda_t [B_t-(1+i_t)(B_{t-1}+w_t N_t+D_t-P_t C_t)].$$

The first order conditions are

$$\lambda_t = \frac{\beta^t C_t^{-\sigma} Z_t}{(1+i_t)P_t},$$
$$\lambda_t = \frac{\beta^t N_t^{\varphi} Z_t}{(1+i_t)w_t},$$
$$\lambda_t = (1+i_{t+1})\lambda_{t+1}.$$

The <u>intratemporal optimality condition</u> is:

$$\frac{N_t^{\varphi}}{C_t^{-\sigma}} = \frac{w_t}{P_t}.$$

The <u>intertemporal optimality condition</u> is:

$$\frac{C_t^{-\sigma} Z_t}{(1+i_t)P_t} = \beta(1+i_{t+1})\frac{C_{t+1}^{-\sigma} Z_{t+1}}{(1+i_{t+1})P_{t+1}}.$$

The Euler equation is:

$$C_t^{-\sigma} = \beta\, \mathbb{E}_t \Big[\frac{(1+i_t)}{(1+\pi_{t+1})} C_{t+1}^{-\sigma} \Big(\frac{Z_{t+1}}{Z_t}\Big)\Big].$$

Inflation is defined as

$$\Pi_{t+1} \equiv 1+\pi_{t+1} \equiv \frac{P_{t+1}}{P_t}.$$

Notice that if there is zero inflation, $\pi = 0$ for all t, then

$$C^{-\sigma} = \beta \frac{1+i}{1} C^{-\sigma}(1).$$

This implies that under zero inflation the nominal interest rate is inversely proportional to the discount rate

$$1+i = 1/\beta.$$

The next step is to log–linearize the Euler equation

$$-\sigma \hat{C}_t = (1 \hat{+} i_t) - \mathbb{E}_t\,(1 + \hat{\pi}_{t+1}) - \sigma\, \mathbb{E}_t\, \hat{C}_{t+1} + \mathbb{E}_t\, \hat{Z}_{t+1} - \hat{Z}_t.$$

Note that

$$(1 \hat{+} i_t) = \mathrm{d}\log(1+i_t) \approx \mathrm{d}i_t$$
$$(1 + \hat{\pi}_{t+1}) = \mathrm{d}\log(1+\pi_{t+1}) \approx \mathrm{d}\pi_{t+1} = \pi_{t+1},$$

because there is no inflation, $\pi = 0$. Also note that

$$\begin{aligned} \mathrm{d}\log Z_t &= \mathrm{d}z_t \\ \mathrm{d}\log Z_{t+1} &= \mathrm{d}z_{t+1}, \end{aligned}$$

and

$$\mathbb{E}_t\,\mathrm{d}z_{t+1} = \rho_z\,\mathrm{d}z_t.$$

It follows that

$$\begin{aligned} -\sigma\hat{C}_t &= -\sigma\hat{C}_{t+1} + \mathrm{d}i_t - \pi_{t+1} + (\rho_z - 1)\,\mathrm{d}z_t \\ \hat{C}_t &= \hat{C}_{t+1} - \tfrac{1}{\sigma}(\mathrm{d}i_t - \pi_{t+1}) - \tfrac{1}{\sigma}(\rho_z - 1)\,\mathrm{d}z_t, \end{aligned}$$

where

$$r_{t+1} \equiv i_t - \pi_{t+1}.$$

It follows that

$$\hat{C}_t = \mathbb{E}_t\,\hat{C}_{t+1} - \tfrac{1}{\sigma}(\mathbb{E}_t\,\mathrm{d}r_{t+1}) - \tfrac{1}{\sigma}(\rho_z - 1)\,\mathrm{d}z_t.$$

Note that $Y_t = C_t$, because there is no government spending and the economy is closed. You can solve forward

$$\begin{aligned} \hat{Y}_t &= -\tfrac{1}{\sigma}\,\mathbb{E}_t(\mathrm{d}r_{t+1}) - \tfrac{1}{\sigma}(\rho_z - 1)\,\mathrm{d}z_t - \tfrac{1}{\sigma}\,\mathbb{E}_{t+1}\,\mathrm{d}r_{t+2} - \tfrac{1}{\sigma}(\rho_z - 1)\,\mathbb{E}_t\,\mathrm{d}z_{t+1} + \hat{Y}_{t+2} \\ \hat{Y}_t &= -\tfrac{1}{\sigma}\,\mathbb{E}_t\sum_{j=1}^{\infty}\mathrm{d}r_{t+j} - \tfrac{1}{\sigma}\,\mathbb{E}_t\sum_{j=0}^{\infty}(\rho_z - 1)\,\mathrm{d}z_{t+j} + \lim_{j\to\infty}\mathbb{E}_t\,\hat{Y}_{t+j}. \end{aligned}$$

Note that $\lim_{j\to\infty}\mathbb{E}_t\,\hat{Y}_{t+j} = 0$, because $\mathbb{E}_t\,\mathrm{d}z_{t+j} = \rho_z^j\,\mathrm{d}z_t$. It follows that

$$\begin{aligned} \hat{Y}_t &= -\tfrac{1}{\sigma}\,\mathbb{E}_t\sum_{j=1}^{\infty}\mathrm{d}r_{t+j} - \tfrac{1}{\sigma}(\rho_z - 1)\sum_{j=0}^{\infty}\rho_z^j\,\mathrm{d}z_t \\ \hat{Y}_t &= -\tfrac{1}{\sigma}\,\mathbb{E}_t\sum_{j=1}^{\infty}\mathrm{d}r_{t+j} - \tfrac{1}{\sigma}(\rho_z - 1)\tfrac{1}{1-\rho_z}\,\mathrm{d}z_t. \end{aligned}$$

You can now see that output today is negatively related to interest rates in the future. Thus any deviation from steady-state output is related to long-term interest rates.

$$\hat{Y}_t = -\frac{1}{\sigma}\,\mathbb{E}_t\sum_{j=1}^{\infty}\mathrm{d}r_{t+j} + \frac{1}{\sigma}\,\mathrm{d}z_t.$$

Note that in the case of zero inflation, then

$$\mathrm{d}r_{t+j} = r_{t+j} - r = r_{t+j} - i,$$

and the nominal interest rate is

$$i = \frac{1}{R} - 1.$$

The firm's optimization problem. Each firm i is subject to a production function

$$Y_t(i) = A_t N_t(i)^{\eta},$$

where technology follows an AR(1) process

$$
\begin{aligned}
a_t &= \log A_t \\
a_t &= \rho_a a_{t-1} + \varepsilon_{a,t}.
\end{aligned}
$$

Each firm i has a demand curve

$$Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} Y_t.$$

The aggregate demand of the economy is

$$\int_0^1 Y_t(i)\,\mathrm{d}j = \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} Y_t\,\mathrm{d}j.$$

If firms are allowed to reset their prices every period, then the model is a repeated Blanchard–Kiyotaki model. However, assume that there are nominal rigidities where firms must post a contract price to satisfy demand.

Definition: Taylor Contracts
Taylor contracts are price contracts that last n periods.

Definition: Calvo Contracts
Calvo contracts are price contracts where every period the contract continues with probability θ and terminates with probability $1-\theta$. The probability of price reset is independent of contract duration and the price chosen. It is assumed that draws of θ are i.i.d. and the expected lifetime is $\frac{1}{\theta}$ periods.

Under Calvo contracts, in period t, there are $1-\theta$ firms that reset prices in period t. You can also find that there are $\theta(1-\theta)$ firms in period t that reset prices in period $t-1$, and $\theta^j(1-\theta)$ firms that reset prices in period $t-j$.

If a firm resets its price, then the optimal price, $P_t(i)$, maximizes profits today and expected in the future. The profit maximization problem is

$$\max_{\{P_t((i)\}} \pi = \sum_{j=0}^{\infty} \underset{\substack{\uparrow \\ \text{Probability} \\ \text{still in effect}}}{\theta^j} \underset{\substack{\uparrow \\ \text{Discount factor}}}{m_{t,t+j}} \frac{1}{P_{t+j}} \underset{\substack{\uparrow \\ \text{Nominal profits in } t+j}}{[P_t(i)Y_{t+j}(i) - w_{t+j}N_{t+j}(i)]}.$$

The stochastic discount factor of the owners of the firms is

$$m_{t,t+j} = \frac{\beta^j C_{t+j}^{-\sigma}}{C_t^{-\sigma}}.$$

Note that

$$Y_{t+j}(i) = \left(\frac{P_t(i)}{P_{t+j}}\right)^{-\epsilon} Y_{t+j}.$$

So,

$$N_{t+j}(i) = \left(\frac{Y_{t+j}(i)}{A_{t+j}}\right)^{\frac{1}{\eta}}.$$

Note that

$$\frac{\partial \log Y_{t+j}(i)}{\partial \log P_t(i)} = -\epsilon.$$
$$\frac{\partial Y_{t+j}(i)}{\partial P_t(i)} = -\epsilon \frac{Y_{t+j}(i)}{P_t(i)}.$$

Also remember that $C_{t+j} = Y_{t+j}$, because there is no government spending or investment in capital. A firm's optimization problem can be written as

$$\max_{\{P_t(i)\}} \pi = \sum_{j=0}^{\infty} \theta^j m_{t,t+j} \tfrac{1}{P_{t+j}} [P_t(i) Y_{t+j}(i) - w_{t+j} Y_{t+j}(i)^{\frac{1}{\eta}} A_{t+j}^{-\frac{1}{\eta}}]$$
$$\text{s. t. } Y_{t+j}(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} Y_{t+j}$$
$$Y_{t+j}(i) = A_{t+j}(i) N_{t+j}(i)^{\eta}.$$

The first–order condition is

$$\frac{\partial \pi}{\partial P_t(i)} = \sum_{j=0}^{\infty} \theta^j m_{t,t+j} \tfrac{1}{P_{t+j}} \left[Y_{t+j}(i) + P_t(i) \left(-\epsilon \frac{Y_{t+j}(i)}{P_t(i)}\right) - w_{t+j} \tfrac{1}{\eta} \left(Y_{t+j}(i)\right)^{\frac{1}{\eta}-1} \left(-\epsilon \frac{Y_{t+j}(i)}{P_t(i)} A_{t+j}^{-\frac{1}{\eta}}\right)\right] = 0.$$

You can then solve for the optimal reset price

$$\sum_{j=0}^{\infty} \theta^j m_{t,t+j} \tfrac{1}{P_{t+j}} [1-\epsilon) Y_{t+j}(i) + \tfrac{\epsilon}{\eta} w_{t+j} A_{t+j}^{-\frac{1}{\eta}} \tfrac{1}{P_t(i)} Y_{t+j}(i)^{\frac{1}{\eta}}] = 0$$
$$\sum_{j=0}^{\infty} \theta^j m_{t,t+j} \tfrac{1}{P_{t+j}} [(1-\epsilon) Y_{t+j}(i) P_t(i) + \tfrac{\epsilon}{\eta} w_{t+j} A_{t+j}^{-\frac{1}{\eta}} Y_{t+j}(i)^{\frac{1}{\eta}}] = 0$$
$$\sum_{j=0}^{\infty} \theta^j m_{t,t+j} \tfrac{1}{P_{t+j}} (1-\epsilon) Y_{t+j}(i) P_t(i) = -\sum_{j=0}^{\infty} \theta^j m_{t,t+j} \tfrac{1}{P_{t+j}} \tfrac{\epsilon}{\eta} w_{t+j} A_{t+j}^{-\frac{1}{\eta}} Y_{t+j}(i)^{\frac{1}{\eta}}$$

$$P_t(i) = -\frac{\frac{\epsilon}{\eta} \sum_{j=0}^{\infty} \theta^j m_{t,t+j} \frac{1}{P_{t+j}} w_{t+j} A_{t+j}^{-\frac{1}{\eta}} Y_{t+j}^{\frac{1}{\eta}}}{(1-\epsilon) \sum_{j=0}^{\infty} \theta^j m_{t,t+j} \frac{1}{P_{t+j}} Y_{t+j}(i)}$$
$$P_t(i) = \frac{\epsilon}{\epsilon - 1} \frac{\sum_{j=0}^{\infty} \theta^j m_{t,t+j} \frac{1}{P_{t+j}} Y_{t+j}(i) w_{t+j} Y_{t+j}^{\frac{1}{\eta}-1} A_{t+j}^{-\frac{1}{\eta}} \frac{1}{\eta}}{\sum_{j=0}^{\infty} \theta^j m_{t,t+j} \frac{1}{P_{t+j}} Y_{t+j}(i)}$$
$$P_t(i) = \frac{\epsilon}{\epsilon - 1} \frac{\sum_{j=0}^{\infty} \theta^j m_{t,t+j} \frac{1}{P_{t+j}} Y_{t+j}(i) \frac{w_{t+j}}{\eta Y_{t+j}(i)^{(\eta-1)/\eta} A_t^{1/\eta}}}{\sum_{j=0}^{\infty} \theta^j m_{t,t+j} \frac{1}{P_{t+j}} Y_{t+j}(i)}.$$

The above equation represents a firm's monopoly markup over its marginal cost. Notice that the total cost of production for a firm in period t is

$$\mathrm{TC}_t = w_t N_t,$$

where

$$N_t(i) = \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{\eta}}.$$

The marginal cost for a firm at time t is

$$\mathrm{MC}_t = \frac{w_t}{\eta Y_t(i)^{(\eta-1)/\eta} A_t^{1/\eta}},$$

and the marginal product of labor at time t is

$$\mathrm{MPL}_t = \eta Y_t(i)^{(\eta-1)/\eta} A_t^{1/\eta}.$$

You can also interpret the optimal price as a markup over the weighted current and expected future marginal costs

$$P_t(i) = \frac{\epsilon}{\epsilon-1} \frac{\sum_{j=0}^{\infty} \theta^j \beta^j \frac{Y_{t+j}^{-\sigma}}{Y_t^{-\sigma}} \frac{1}{P_{t+j}} w_{t+j} \frac{1}{\eta} (\frac{P_t(i)}{P_{t+j}})^{-\epsilon/\eta} Y_{t+j}^{1/\eta} A_{t+j}^{-1/\eta}}{\sum_{j=0}^{\infty} \theta^j \beta^j \frac{Y_{t+j}^{-\sigma}}{Y_t^{-\sigma}} \frac{1}{P_{t+j}} (\frac{P_t(i)}{P_{t+j}})^{-\epsilon} Y_{t+j}}.$$

Any variable that does not depend on j can be brought out of the summation and canceled

$$\begin{aligned}
P_t(i) &= \frac{\epsilon}{\epsilon-1} \frac{P_t(i)^{-\epsilon/\eta}}{P_t(i)^{-\epsilon}} \frac{\sum_{j=0}^{\infty} \theta^j \beta^j Y_{t+j}^{-\sigma} \frac{1}{P_{t+j}} w_{t+j} \frac{1}{\eta} P_{t+j}^{\epsilon/\eta} Y_{t+j}^{1/\eta} A_{t+j}^{-1/\eta}}{\sum_{j=0}^{\infty} \theta^j \beta^j Y_{t+j}^{-\sigma} \frac{1}{P_{t+j}}} \\
P_t(i)^{1+\epsilon(\frac{1-\eta}{\eta})} &= \frac{\epsilon}{\epsilon-1} \frac{\sum_{j=0}^{\infty} \theta^j \beta^j Y_{t+j}^{1/\eta-\sigma} P_{t+j}^{\epsilon/\eta-1} w_{t+j} A_{t+j}^{-1/\eta} \frac{1}{\eta}}{\sum_{j=0}^{\infty} \theta^j \beta^j Y_{t+j}^{1-\sigma} P_{t+j}^{\epsilon-1}} \frac{P_t^{1-\epsilon}}{P_t^{1-\epsilon}} \\
P_t(i)^{1+\epsilon(\frac{1-\eta}{\eta})} &= \frac{\epsilon}{\epsilon-1} \frac{\sum_{j=0}^{\infty} \theta^j \beta^j Y_{t+j}^{1-\sigma} Y_{t+j}^{1/\eta-1} (\frac{P_{t+j}}{P_t})^{\epsilon/\eta} \frac{w_{t+j}}{P_{t+j}} \frac{1}{\eta} A_{t+j}^{-1/\eta} P_t^{1+\epsilon((1-\eta)/\eta)}}{\sum_{j=0}^{\infty} \theta^j \beta^j Y_{t+j}^{1-\sigma} (\frac{P_{t+j}}{P_t})^{\epsilon-1}} \\
\left(\frac{P_t(i)}{P_t}\right)^{1+\epsilon(\frac{1-\eta}{\eta})} &= \frac{\epsilon}{\epsilon-1} \frac{\sum_{j=0}^{\infty} \theta^j \beta^j Y_{t+j}^{1-\sigma} (\frac{P_{t+j}}{P_t})^{\epsilon/\eta} \frac{w_{t+j}/P_{t+j}}{\eta Y_{t+j}((\eta-1)/\eta) A_{t+j}^{1/\eta}}}{\sum_{j=0}^{\infty} \theta^j \beta^j Y_{t+j}^{1-\sigma} (\frac{P_{t+j}}{P_t})^{\epsilon-1}}.
\end{aligned}$$

A firm's real optimal price, relative to other prices in the economy, is a markup over the economies average expected marginal cost.

You can let

$$\left(\frac{P_t(i)}{P_t}\right)^{1+\epsilon(\frac{1-\eta}{\eta})} = \frac{\epsilon}{\epsilon-1} \frac{Z_{n,t}}{Z_{d,t}}, \tag{4}$$

where

$$\begin{aligned}
Z_{n,t} &= Y_t^{1-\sigma} \frac{w_t/P_t}{\eta Y_t^{(\eta-1)/\eta} A_t^{1/\eta}} + \beta\theta\, \mathbb{E}_t[\Pi_{t+1}^{\epsilon/\eta} Z_{n,t+1}] \\
Z_{d,t} &= Y_t^{1-\sigma} + \beta\theta\, \mathbb{E}_t[\Pi_{t+1}^{\epsilon-1} Z_{d,t+1}],
\end{aligned}$$

and

$$\Pi_t \equiv \frac{P_{t+1}}{P_t}.$$

You can check

$$Z_{d,t} = \sum_{j=0}^{\infty} \theta^j \beta^j Y_{t+j}^{1-\sigma} (\tfrac{P_{t+j}}{P_t})^{\epsilon-1}.$$

A firm's marginal cost is equal to the real wage to marginal product of labor ratio

$$\text{MC}_t = \frac{w_t/P_t}{\text{MPL}_t}.$$

The representative households set their trade–off between consumption and leisure such that

$$\text{MRS}_t = w.$$

Note that in equilibrium

$$\tfrac{\epsilon}{\epsilon-1}\text{MC}_t = 1 = \tfrac{\epsilon}{\epsilon-1}\frac{w_t/P_t}{\text{MPL}_t}.$$

So,

$$\frac{w_t}{P_t} < \text{MPL}_t,$$

because $\text{MC}_t < 1$. The monopoly distortion results in a real wage less than the marginal product of labor

$$w_t < \text{MPL}_t.$$

Also, note that the marginal rate of transformation is

$$\text{MRT} > \frac{w_t}{P_t}.$$

So, the economy can trade–off an increase in the real wage for more production, but the households have reached their optimal point where

$$\text{MRS}_t = \frac{w_t}{P_t}.$$

Therefore, there is an aggregate demand externality, because of underproduction

$$\text{MRT}_t > \text{MRS}_t.$$

The households are currently maximizing utility and are not better off from a transition. Even if the agents owned the shares of the firms, there is still no one firm that wants to change production first and aggregate coordination may not be possible.

If you assume a zero inflation steady–state, $P_t \to P$, $Y_t \to Y$, $w_t \to w$, and $\Pi \to 1$, then

$$\begin{aligned} Z_n &= Y^{1-\sigma}\frac{w/P}{\eta Y^{(\eta-1)/\eta}A^{1/\eta}} + \beta\theta Z_n \\ Z_d &= Y^{1-\sigma} + \beta\theta Z_d. \end{aligned}$$

The steady state values are

$$\begin{aligned} Z_n &= \frac{1}{1-\beta\theta}\left(Y^{1-\sigma}\frac{w/P}{\eta Y^{(\eta-1)/\eta}A^{1/\eta}}\right) \\ Z_d &= \frac{1}{1-\beta\theta}Y^{1-\sigma}. \end{aligned}$$

Recall that

$$\begin{aligned} P_t &= \left(\int_0^1 P_t(i)^{1-\epsilon}\,\mathrm{d}_i\right)^{1/(1-\epsilon)} \\ P_t^{1-\epsilon} &= \int_0^1 P_t(i)^{1-\epsilon}. \end{aligned}$$

Thus,

$$P_t^{1-\epsilon} = \underset{\substack{\uparrow \\ \text{Porportion that} \\ \text{reset in period } t}}{(1-\theta){P_t^*}^{1-\epsilon}} + \underset{\substack{\uparrow \\ \text{Porportion that} \\ \text{reset last period}}}{\theta(1-\theta){P_{t-1}^*}^{1-\epsilon}} + \cdots + \underset{\substack{\uparrow \\ \text{Porportion that} \\ \text{reset in period } j}}{\theta^j(1-\theta){P_{t-j}^*}^{1-\epsilon}} + \ldots$$

Note that all firms choose the same reset price if they can reset. Also, the summation of past prices governs yesterday's price, $P_{t-1}^{1-\epsilon}$, so

$$P_t^{1-\epsilon} = (1-\theta){P_t^*}^{1-\epsilon} + \theta P_{t-1}^{1-\epsilon}.$$

It follows

$$\begin{aligned} \left(\frac{P_t}{P_{t-1}}\right)^{1-\epsilon} &= (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{1-\epsilon}\left(\frac{P_t}{P_{t-1}}\right)^{1-\epsilon} + \theta \\ \Pi_t^{1-\epsilon} &= (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{1-\epsilon}\Pi_t^{1-\epsilon} + \theta. \end{aligned}$$

In the zero inflation steady–state

$$1 = (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{1-\epsilon} + \theta\Pi_t^{\epsilon-1}.$$

If you log–linearize, then

$$0 = \frac{(1-\theta)}{(1-\theta)+\theta}(1-\epsilon)[\hat{P}_t^* - \hat{P}_t] + \frac{\theta}{(1-\theta)+\theta}(\epsilon-1)\hat{\Pi}_t.$$

This relates a firm's reset price to inflation

$$\hat{\Pi}_t = \frac{1-\theta}{\theta}(\hat{P}_t^* - \hat{P}_t).$$

By log–linearizing the equation

$$\left(\frac{P_t(i)}{P_t}\right)^{1+\epsilon(\frac{1-\eta}{\eta})} = \frac{\epsilon}{\epsilon-1}\frac{Z_{n,t}}{Z_{d,t}}, \tag{5}$$

you can determine if the chosen reset price has a positive or negative effect on inflation

$$(1+\epsilon^{(1-\eta)/\eta})(\hat{P}_t^* - \hat{P}_t) = \hat{Z}_{n,t} - \hat{Z}_{d,t},$$

where

$$\begin{aligned} \hat{Z}_{n,t} &= (1-\beta\theta)[(1-\sigma)\hat{Y}_t + \hat{w}_t - \hat{P}_t - \tfrac{\eta-1}{\eta}\hat{Y}_t - \tfrac{1}{\eta}\hat{A}_t] + \beta\theta(\tfrac{\epsilon}{\eta}\hat{\Pi}_{t+1} + \hat{Z}_{n,t+1}) \\ \hat{Z}_{d,t} &= (1-\beta\theta)(1-\sigma)\hat{Y}_t + \beta\theta[(\epsilon-1)\hat{\Pi}_{t+1} + \hat{Z}_{d,t+1}. \end{aligned}$$

It follows that

$$\begin{aligned} \hat{Z}_{n,t} - \hat{Z}_{d,t} &= (1-\beta\theta)[\hat{w}_t - \hat{P}_t - \tfrac{\eta-1}{\eta}\hat{Y}_t - \tfrac{1}{\eta}\hat{A}_t] + \beta\theta[(1-\epsilon+\tfrac{\epsilon}{\eta})\hat{\Pi}_{t+1} + \hat{Z}_{n,t+1} - \hat{Z}_{d,t+1} \\ (1+\epsilon(\tfrac{1-\eta}{\eta}))\tfrac{\theta}{1-\theta}\hat{\Pi}_t &= (1-\beta\theta)[\hat{w}_t - \hat{P}_t - \tfrac{\eta-1}{\eta}\hat{Y}_t - \tfrac{1}{\eta}\hat{A}_t] + \beta\theta[(1+\epsilon(\tfrac{1-\eta}{\eta}))\hat{\Pi}_{t+1}] + \beta\theta((1+\epsilon(\tfrac{1-\eta}{\eta}))\tfrac{\theta}{1-\theta}\hat{\Pi}_{t+1} \end{aligned}$$

and

$$\hat{\Pi}_t = \beta\,\mathbb{E}_t\,\hat{\Pi}_{t+1} + \frac{(1-\beta\theta)(1-\theta)}{\theta}\frac{1}{1+\epsilon((1-\eta)/\eta)}\left[\hat{w}_t - \hat{P}_t - \frac{\eta-1}{\eta}\hat{Y}_t - \frac{1}{\eta}\hat{A}_t\right].$$

If marginal costs are high, then firms will raise price. Conversely, if marginal costs are low, then firms will lower price. Also, if firms expect inflation to be higher in the future, then their optimal reset price will be higher. Alternatively, if firms expect inflation to be lower in the future, then their optimal prices will be lower.

You can relate the output gap to inflation by defining a natural frictionless level of output

$$\hat{Y}_{n,t}.$$

The natural level of output is the level of output firms would chose in a frictionless economy without price stickiness. From the representative household's optimization problem, you can find that

$$\hat{w}_{n,t} - \hat{P}_{n,t} = \varphi\hat{N}_{n,t} + \sigma\hat{Y}_{n,t}.$$

Marginal rate
of substitution

The competitive labor market implies that

$$\begin{aligned}
\hat{w}_{n,t}P_{n,t} &= \frac{\epsilon}{\epsilon-1}\mathrm{MPL}_t \\
\hat{w}_{n,t} - \hat{P}_{n,t} &= \hat{Y}_{n,t} - \hat{N}_{n,t}.
\end{aligned}$$

If you log–linearize the production function you find

$$\hat{Y}_{n,t} = \varphi\hat{N}_{n,t} + \hat{A}_t.$$

It follows that

$$\begin{aligned}
\varphi\hat{N}_{n,t} + \sigma\hat{Y}_{n,t} &= \hat{Y}_{n,t} - \hat{N}_{n,t} \\
(1-\sigma)\hat{Y}_{n,t} &= (1+\varphi)\hat{N}_{n,t} \\
(1-\sigma)\hat{Y}_{n,t} &= (1+\varphi)\tfrac{1}{\eta}(\hat{Y}_{n,t} - \hat{A}_t) \\
[(1-\sigma) - \tfrac{1+\varphi}{\eta}]\hat{Y}_{n,t} &= -\tfrac{1+\varphi}{\eta}\hat{A}_t \\
\hat{Y}_{n,t} &= \frac{(1+\varphi)/\eta}{(1+\varphi)/\eta - (1-\sigma)}\hat{A}_{n,t}.
\end{aligned}$$

The frictionless level of output can be written as

$$\hat{Y}_{n,t} = \frac{1}{1-(1-\sigma)\eta/(1+\varphi)}\hat{A}_t.$$

If there is a positive technology shock, then both output and labor increase. Conversely, if there is a negative technology shock, then both output and labor decrease. You can solve for the frictionless real wage and labor hours as well.

Next, note that

$$\begin{aligned}
\hat{w}_t - \hat{P}_t &= \varphi \hat{N}_t + \sigma \hat{Y}_t \\
\hat{w}_t - \hat{P}_t - \tfrac{\eta-1}{\eta}\hat{Y}_t - \tfrac{1}{\eta}\hat{A}_t &= \varphi \hat{N}_t + \sigma \hat{Y}_t - \tfrac{\eta-1}{\eta}\hat{Y}_t - \tfrac{1}{\eta}\hat{A}_t \\
\hat{w}_t - \hat{P}_t - \tfrac{\eta-1}{\eta}\hat{Y}_t - \tfrac{1}{\eta}\hat{A}_t &= \tfrac{\varphi}{\eta}(\hat{Y}_t - \hat{A}_t) + \sigma \hat{Y}_t - \tfrac{\eta-1}{\eta}\hat{Y}_t - \tfrac{1}{\eta}\hat{A}_t \\
\hat{w}_t - \hat{P}_t - \tfrac{\eta-1}{\eta}\hat{Y}_t - \tfrac{1}{\eta}\hat{A}_t &= [\sigma + \tfrac{\varphi+1-\eta}{\eta}]\hat{Y}_t - \tfrac{1+\varphi}{\eta}\hat{A}_t \\
\hat{w}_t - \hat{P}_t - \tfrac{\eta-1}{\eta}\hat{Y}_t - \tfrac{1}{\eta}\hat{A}_t &= [\sigma + \tfrac{\varphi+1-\eta}{\eta}]\hat{Y}_t - [\tfrac{1+\varphi}{\eta} - (1-\sigma)]\hat{Y}_{n,t}
\end{aligned}$$

and

$$\hat{\Pi}_t = \beta\, \mathbb{E}_t\, \hat{\Pi}_{t+1} + \tfrac{(1-\beta\theta)(1-\theta)}{\theta} \tfrac{1}{1+\epsilon(1-\eta)/\eta}(\hat{w}_t - \hat{P}_t - \tfrac{\eta-1}{\eta}\hat{Y}_t - \tfrac{1}{\eta}\hat{A}_t).$$

The New Keynesian Phillip's Curve is

$$\hat{\Pi}_t = \beta\, \mathbb{E}_t\, \hat{\Pi}_{t+1} + \gamma(\hat{Y}_t - \hat{Y}_{n,t}),$$

where

$$\gamma = \frac{(1-\beta\theta)(1-\theta)}{\theta} \frac{1}{1+\epsilon(1-\eta)/\eta}\left(\sigma + \frac{\varphi+1-\eta}{\eta}\right).$$

If you define

$$X_t \equiv \hat{Y}_t - \hat{Y}_{n,t},$$

then you can write the log–linearized gross inflation rate as

$$\hat{\Pi}_t = \beta\, \mathbb{E}_t\, \hat{\Pi}_{t+1} + \gamma X_t,$$

where

$$\hat{\Pi}_t = (1 \hat{+} \pi_t) \approx \pi_t.$$

Furthermore, from the equation for deviation from steady state output, you can find that

$$\hat{Y}_{n,t} = \mathbb{E}_t\, \hat{Y}_{n,t+1} - \tfrac{1}{\sigma}\, \mathbb{E}_t\, r_{n,t+1} + \varepsilon_{y,t}.$$

So, you can write the New Keynesian Investment–Savings (IS) Curve

$$\hat{Y}_t - \hat{Y}_{n,t} = \mathbb{E}_t(\hat{Y}_{t+1} - \hat{Y}_{n,t+1}) - \tfrac{1}{\sigma}\, \mathbb{E}_t(r_{t+1} - r_{n,t+1}).$$

Note that the shock to the households' preference for waiting is

$$\varepsilon_{y,t} = \tfrac{1-\rho_z}{\sigma} Z_t.$$

Approximations in the model are accurate for small percent deviations from steady–state. Given the exogenous preference shock to Z_t and the exogenous technology shock to A_t, the central bank sets policy to influence the nominal interest rate as a function of the realized output gap, X_t, and the rate of inflation, π_t.

3.6.2. Rational Expectations

Definition: The Taylor Rule
The Taylor Rule states that the central bank sets the nominal interest rate

$$i_t = \bar{i} + a x_t + b\pi_t + \varepsilon_{i,t},$$

where $a > 0$ and $b > 0$ are the bank's chosen responses to the output gap, x_t, and inflation, Π_t, respectively. You can then convert the nominal interest rate to the real interest rate

$$r_{t+1} = i_t - \pi_{t+1}.$$

The main idea is that household will have rational expectations for future interest rates

$$\mathbb{E}_t\, r_{t+1} = i_t - \mathbb{E}_t\, \pi_{t+1}.$$

The New Keynesian DSGE model results in a system of 3 equations

$$\begin{aligned} x_t &= \mathbb{E}_t\, x_{t+1} - \tfrac{1}{\sigma}\,\mathbb{E}_t(r_{t+1} - r_{n,t+1}) \\ \pi_t &= \beta\,\mathbb{E}_t\, \pi_{t+1} + \gamma x_t + \varepsilon_{\pi,t} \\ i_t &= a x_t + b\pi_t + \varepsilon_{i,t}. \end{aligned}$$

Assuming that the central bank can optimally choose a and b, you can solve the system using the Blanchard–Kahn method. You can write the system in Blanchard–Kahn form with 2 variables, x_t and π_t, using the Taylor rule, $r_{t+1} = i_t - \pi_{t+1}$ and $\mathbb{E}_t\, r_{t+1} = i_t - E_t\pi_{t+1}$. In order for there to be a unique solution to the system, the eigenvalues must be of an appropriate magnitude. That is because x_t and π_t are both jump variables. The eigenvalues of the Blanchard–Kahn matrix Λ must be greater than one in absolute value, $|\lambda_x| > 1$ and $|\lambda_\pi| > 1$. That is, you need to have explosive roots. The solution will jump to the optimal value each period. Some values of (a, b) can violate eigenvalue requirements. Note that both eigenvalues are explosive, $|\lambda_x| > 1$ and $|\lambda_\pi| > 1$, if and only if $b > 1$. Thus, if the central bank does not respond strongly to inflation, then the model will not have a unique stable equilibrium and there will be many possible equilibria.

Definition: The Taylor Principle
The Taylor Principle implies that good monetary policy should set $b > 1$.

Arguably, good fiscal policy will influence the real interest rate to fight inflation and let the real interest rate respond positively to inflation. If not, then there will be multiple equilibria. If $b < 1$, then there is 1 or no explosive eigenvalue and at least 1 nonexplosive eigenvalue. Therefore, there will be many nonexplosive solutions. This allows the central bank to make any policy choice and the economy will settle down. This is because (a, b) are not pinned down and may fluctuate freely if $b < 1$.

Another caveat of the Taylor Rule is that there is an optimal steady–state rate of inflation. Note that a policymaker can try to set a positive output gap, $x_t > 0$, because of the monopolistic distortion. However, a higher output gap, x_t, will lead to a higher rate of inflation, π_t. If a positive output gap is always set, $x_t = x > 0$, then the rate of inflation is

$$\begin{aligned} \pi &= \beta\,\mathbb{E}_t\, \pi + \gamma x \\ (1-\beta)\pi &= \gamma x \\ \pi &= \frac{1}{1-\beta}\gamma x. \end{aligned}$$

Note that the rate of inflation π does not explode. The central bank can implement a positive output gap, $x > 0$, without the economy exploding. However, an increase in the rate of inflation, π, leads to an increase in the relative price dispersion, Δ. The central bank can try to alleviate either the monopolistic or price distortion, but will increase the other. That is, there is a trade–off between monopolistic distortion and price distortion. Thus, the central bank can find an optimal balance between a positive output gap and the level of relative price dispersion.

For the central bank to choose the optimal policy (a^*, b^*), then it needs to determine the best potential outcome based on some criterion. You can hypothesize that the policy maker has a loss function. Assume that the central banks desires to keep inflation low and output close to trend. In this case, the central bank desires to minimize variation in x_t and π_t. The loss function is

$$L = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \phi x_t^2).$$

Given exogenous parameters, σ and γ, the central bank chooses a and b to minimize their loss function. This is a linear–quadratic optimization problem. Thus, all the first–order conditions are linear equations and the optimal policy can be solved for. Empirically, a^* and b^* are usually large. The optimization problem is

$$\begin{aligned} \min_{\{a,b\}} L &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \phi x_t^2) \\ \text{s. t. } x_t &= \mathbb{E}_t\, x_{t+1} - \tfrac{1}{\sigma}\, \mathbb{E}_t[i_t - \pi_{t+1} - r_{n,t+1}] \\ \pi_t &= \beta\, \mathbb{E}_t\, \pi_{t+1} + \gamma x_t + \varepsilon_{\pi_t}. \end{aligned}$$

The Lagrangian for the problem is

$$\min_{\{x_t,\pi_t,i_t\}} \mathscr{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \phi x_t^2) - \lambda_{1,t}(x_t - \mathbb{E}_t\, x_{t+1} + \tfrac{1}{\sigma}\, \mathbb{E}_t[i_t - \pi_{t+1} - r_{n,t+1}]) - \lambda_{2,t}(\pi_t - \beta\, \mathbb{E}_t\, \pi_{t+1} - \gamma x_t - \varepsilon_{\pi,t}),$$

where

$$\mathbb{E}_t\, x_{t+1} = \sum_{s=1}^{S} \frac{P_{s,t+1}}{P_{s,t}} x_{t+1}.$$

You can then solve for the optimal nominal interest rate $i_{s,t}$ at each time t and state s_t. The first order conditions are

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial x_t} &= 0 \\ \frac{\partial \mathscr{L}}{\partial \pi_t} &= 0 \\ \frac{\partial \mathscr{L}}{\partial i_t} &= 0. \end{aligned}$$

There is an optimal commitment solution, because the model is completely specified. However, the solution is not time–consistent. Normally, in dynamic programming, if you reoptimize at a future date then then the new optimal plan is consistent with the old optimal plan according to Bellman's optimality principle. However, here there are constraints that depend on expectations. If the policy maker wants to decrease the rate of inflation, π_t, then there are two main effects.

- A decrease in the rate of inflation, π_t,, results in a negative output gap, x_t.
- By promising a negative output gap, x_{t+j}, then there will be a lower expected future rate of inflation, $E_t \pi_{t+1}$.

The central bank could attempt to spread the cost of disinflation across time. If there is no commitment technology, then firms and households will not believe the plan and it cannot be achieved. If the policy make reoptimizes at a future date, then a different choice in policy will be made.

If you assume that the policy maker dislikes changing the interest rate quickly, perhaps because of an adjustment cost. The loss function is then

$$L = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \phi x_t^2 + \omega(\Delta i_t)^2),$$

where θ and ω represent the central bank's preferences. Empirically, this loss function produces more reasonable policy choices (a^*, b^*). However, the problem with this approach is that it is ad hoc. The Federal Open Market Committee (OMC) uses this model in practice.

Another method to produce cautionary policy is to introduce parameter uncertainty. If you are uncertain about exogenous parameters then you introduce volatility from policy choice. Large change in i_t and r_t will lead to larger variances in x_t and π_t.

Another policy could be to choose (a, b) to maximize the households' welfare

$$\max_{\{a,b\}} U = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{N_t^{1-\varphi}}{1-\varphi} \right] Z_t.$$

The major drawback is that the loss function is no longer linear–quadratic. Furthermore, Gorman aggregation does not apply to welfare aggregation. A possible solution could be to find a second–order approximation for the households' welfare.

You can try various other interest rate rules. You can assume that the central bank desires to find the best $AR(p)$ process

$$i_t = \rho_1 i_t t - 1 + \cdots + \rho_p + a x_t + b \pi_t + \varepsilon_{i,t}.$$

Alternatively, the central bank could set the interest rate based on future expectations

$$i_t = a \, \mathbb{E}_t \, x_{t+1} + b \, \mathbb{E}_t \, \pi_{t+1} + \varepsilon_{i,t}.$$

You could compare different rules to find which performs best empirically.

Kidlen and Prescrott note that if you have no control over $\mathbb{E}_t \, \pi_{t+1}$, then there is lack of commitment to a chosen policy and there will be lack of belief from firms and households. The first–order conditions of the policymaker's optimization problem are different from the commitment solution. Nevertheless, in equilibrium you can set $\pi_t^e = \mathbb{E}_t \, \pi_{t+1}$. The policymaker cannot affect expectations of inflation, π_t^e, or the output gap, x_t^e. Therefore, rational expectations are present in equilibrium. There is an optimal discretion solution that is time consistent. There are no longer expectations that can be changed by promises of the policymaker. Note that the optimal discretion solution is necessarily worse than the optimal commitment solution that would be chosen if the policymaker could follow through on promises.

3.6.3. Price Dispersion in the New Keynesian DSGE Model

The New Keynesian DSGE model results in price dispersion where there are a range of firms, some with high marginal costs and others with low marginal costs. The production function of firm i is

$$Y_t(i) = A_t N_t(i)^\eta.$$

The firm's marginal cost is

$$\mathrm{MC}_t(i) = \frac{w_t/P_t}{\mathrm{MPL}_t(i)},$$

where the firm's marginal product of labor is

$$\mathrm{MPL}_t(i) \equiv \eta \frac{Y_t(i)}{N_t(i)}.$$

The average marginal product of labor in the economy is

$$\mathrm{MPL}_t \approx \eta \frac{Y_t}{N_t}.$$

Proof. The labor employed by firm i is

$$N_t(i) = \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{\eta}}.$$

The total labor employed in the economy is

$$N_t = \int_0^1 N_t(i)\,\mathrm{d}i.$$

It follows that

$$N_t = \int_0^1 Y_t(i)^{\frac{1}{\eta}} A_t^{-\frac{1}{\eta}}\,\mathrm{d}i.$$

If you substitute for firm i's demand, then

$$N_t = A_t^{-\frac{1}{\eta}} \int_0^1 \left[\left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} Y_t\right]^{\frac{1}{\eta}} \mathrm{d}i$$

$$N_t = A_t^{-\frac{1}{\eta}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\epsilon}{\eta}} Y_t^{\frac{1}{\eta}}\,\mathrm{d}i$$

$$N_t = Y_t^{\frac{1}{\eta}} A_t^{-\frac{1}{\eta}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\epsilon}{\eta}} \mathrm{d}i.$$

Thus,

$$Y_t = A_t N_t^\eta \left[\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\epsilon}{\eta}} \mathrm{d}i\right]^{-\eta},$$

Measure of price dispersion

If price dispersion is low, then

$$\mathrm{MPL}_t = \frac{\partial Y_t}{\partial N_t} \approx \eta \frac{Y_t}{N_t}.$$

□

Define the level of price dispersion as

$$\Delta \equiv \left[\int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\frac{\epsilon}{\eta}} \right]^{\eta}.$$

Note that when there is more heterogeneity in prices the economy produces less. You can write the production function as

$$Y_t = A_t \Delta_t^{-1} N_t^{\eta}.$$

Claim: The level of price dispersion must be greater than or equal to unity

$$\Delta \geq 1.$$

Proof. Consider a convex function

$$f(x) = \left(x^{\frac{\epsilon}{\epsilon - 1}} \right)^{\frac{1}{\eta}},$$

where $\frac{\epsilon}{\epsilon-1} > 1$, $\frac{1}{\eta} > 1$, and $\eta \in (0, 1)$. By Jensen's inequality, a convex function of an integral is less than or equal to the integral applied after the convex transformation

$$\underbrace{\int_0^1 \left(x^{\frac{\epsilon}{\epsilon-1}} \right)^{\frac{1}{\eta}} \mathrm{d}i}_{\text{Average of the function}} \geq \underbrace{\left[\int_0^1 x_i \, \mathrm{d}i \right]^{\frac{\epsilon}{\epsilon-1} \frac{1}{\eta}}}_{\text{Function of the average}}.$$

If you let

$$x(i) = \left(\frac{P_t(i)}{P_t} \right)^{1-\epsilon},$$

then

$$\int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\frac{\epsilon}{\eta}} \mathrm{d}i \geq \left[\int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{1-\epsilon} \mathrm{d}i \right]^{\frac{\epsilon}{\epsilon-1} \frac{1}{\eta}}.$$

Recall that

$$P_t \equiv \int_0^1 P_t(i)^{1-\epsilon} \, \mathrm{d}i.$$

It follows that

$$\int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\frac{\epsilon}{\eta}} \mathrm{d}i \geq 1.$$

Thus,

$$\Delta \equiv \left[\int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\frac{\epsilon}{\eta}} \right]^{\eta} \geq 1.$$

□

The level of price dispersion is unity, $\Delta_t = 1$, if and only if all firms set the same price, $P_t(i) = P_t$ for all firms i. The socially optimal level of output is where

$$Y_t = A_t N_t^{\eta}.$$

In general,

$$Y_t < A_t N_t^{\eta},$$

because there are monopolistic distortions and there is relative price dispersion, $\Delta_t > 1$.

Claim: Price dispersion, Δ_t, is a second–order term near the zero inflation steady–state.

The idea is that at the zero inflation steady–state, then $\Delta = 1$. If there is a positive or negative shock to the economy, then the first–order effect is $\mathrm{d}\Delta = 0$.

Proof. First, you can write the level of price dispersion

$$\Delta_t^{\frac{1}{\eta}} = \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\epsilon}{\eta}},$$

in recursive form as

$$\Delta_t^{\frac{1}{\eta}} = (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\frac{\epsilon}{\eta}} + \theta(1-\theta)\left(\frac{P_{t-1}^*}{P_t}\right)^{-\frac{\epsilon}{\eta}} + \cdots + \theta^j(1-\theta)\left(\frac{P_{t-j}^*}{P_t}\right)^{-\frac{\epsilon}{\eta}} + \ldots$$

It follows that

$$\Delta_{t-1} = \theta\left(\frac{P_{t-1}^*}{P_t}\right)^{-\frac{\epsilon}{\eta}}(1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{-\frac{\epsilon}{\eta}} + \cdots + \theta^{j-1}(1-\theta)\left(\frac{P_{t-j}}{P_{t-1}}\right)^{-\frac{\epsilon}{\eta}} + \ldots$$

and

$$\begin{aligned}
\Delta_t^{\frac{1}{\eta}} &= (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\frac{\epsilon}{\eta}} + \theta\left(\frac{P_{t-1}}{P_t}\right)^{-\frac{\epsilon}{\eta}}\Delta_{t-1}^{\frac{1}{\eta}} \\
\Delta_t^{\frac{1}{\eta}} &= (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\frac{\epsilon}{\eta}} + \theta\Pi_t^{\frac{\epsilon}{\eta}}\Delta_{t-1}^{\frac{1}{\eta}}.
\end{aligned}$$

If you log–linearize the relation, then

$$\frac{1}{\eta}\hat{\Delta}_t = \frac{1-\theta}{1-\theta+\theta}\left(-\frac{\epsilon}{\eta}\right)\left(\hat{P}_t^* - \hat{P}_t\right) + \frac{\theta}{1-\theta+\theta}\left(\frac{\epsilon}{\eta}\hat{\Pi}_t + \frac{1}{\eta}\hat{\Delta}_{t-1}\right),$$

and

$$\hat{\Delta}_t = (1-\theta)(-\epsilon)(\hat{P}_t^* - \hat{P}_t) + \theta(\epsilon)\hat{\Pi}_t + \theta\hat{\Delta}_{t-1}.$$

Recall that

$$\hat{\Pi} = \frac{1-\theta}{\theta}(\hat{P}_t^* - \hat{P}_t).$$

It follows that

$$\hat{\Delta}_t = (1-\theta)(-\epsilon)(\hat{P}_t^* - \hat{P}_t) + (1-\theta)(\epsilon)(\hat{P}_t^* - \hat{P}_t) + \theta\hat{\Delta}_{t-1},$$

and the level of price dispersion follows an AR(1) process

$$\hat{\Delta}_t = \theta\hat{\Delta}_{t-1}.$$

Thus, first–order changes in price dispersion are affected only by the previous level of price dispersion and not by any current period exogenous shocks. $\square$

3.6.4. Applications of New Keynesian DSGE Models

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3.7. Expectations and Time-Inconsistency

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3.8. Sticky Wages and Prices

Reading: Erceg, Christopher, Dale Henderson, and Andrew Levin (2000). "Optimal Monetary Policy with Staggered Wage and Price Contracts," *Journal of Monetary Economics* 46, 281-313.

Reading: Christiano, Lawrence, Martin Eichenbaum, and Charles Evans (2005). "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy* 113, 1-45.

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3.9. Macroeconomic vs. Microeconomic Elasticities

Reading: Altig, David, Lawrence Christiano, Martin Eichenbaum, and Jesper Lind'e (2010). "Firm-Specific Capital, Nominal Rigidities, and the Business Cycle," *Review of Economic Dynamics* 14, 225-247.